Exact time dependence of solutions to the time-dependent Schrödinger equation

2009 J. Phys. A: Math. Theor. 42 035307

(http://iopscience.iop.org/1751-8121/42/3/035307)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 035307 (33pp) doi:10.1088/1751-8113/42/3/035307

Exact time dependence of solutions to the time-dependent Schrödinger equation

M A Lohe

Department of Physics, The University of Adelaide, 5005, Australia

E-mail: Max.Lohe@adelaide.edu.au

Received 18 September 2008, in final form 17 November 2008
Published 9 December 2008
Online at stacks.iop.org/JPhysA/42/035307

Abstract

Solutions of the Schrödinger equation with an exact time dependence are derived as eigenfunctions of dynamical invariants which are constructed from time-independent operators using time-dependent unitary transformations. Exact solutions and a closed form expression for the corresponding time evolution operator are found for a wide range of time-dependent Hamiltonians in d dimensions, including non-Hermitean $\mathcal{PT}$-symmetric Hamiltonians. Hamiltonians are constructed using time-dependent unitary spatial transformations comprising dilatations, translations and rotations and solutions are found in several forms: as eigenfunctions of a quadratic invariant, as coherent state eigenfunctions of boson operators, as plane wave solutions from which the general solution is obtained as an integral transform by means of the Fourier transform, and as distributional solutions for which the initial wavefunction is the Dirac δ-function. For the isotropic harmonic oscillator in d dimensions radial solutions are found which extend known results for $d = 1$, including Barut–Girardello and Perelomov coherent states (i.e., vector coherent states), which are shown to be related to eigenfunctions of the quadratic invariant by the ζ-transformation. This transformation, which leaves the Ermakov equation invariant, implements SU(1, 1) transformations on linear dynamical invariants. $\mathfrak{su}(1,1)$ coherent states are derived also for the time-dependent linear potential. Exact solutions are found for Hamiltonians with electromagnetic interactions in which the time-dependent magnetic and electric fields are not necessarily spatially uniform. As an example, it is shown how to find exact solutions of the time-dependent Schrödinger equation for the Dirac magnetic monopole in the presence of time-dependent magnetic and electric fields of a specified form.

PACS numbers: 03.65.Ge, 03.65.Ca, 03.65.Fd

1. Introduction

The time-dependent Schrödinger equation (TDSE), $i\hbar\partial\psi(\boldsymbol{x},t)/\partial t = \mathcal{H}(t)\psi(\boldsymbol{x},t)$, where $\mathcal{H}(t)$ is a time-dependent Hermitean Hamiltonian in d dimensions, determines the evolution of the wavefunction ψ from the initial time $t = t_0$ to a later time t according to $\psi(\boldsymbol{x},t) = U(t,t_0)\psi(\boldsymbol{x},t_0)$. The time evolution operator U is unitary, by conservation of probability, and satisfies

$$i\hbar\frac{\partial U}{\partial t} = \mathcal{H}(t)U. \tag{1}$$

Solutions ψ and U exist for conventional Hamiltonians provided the initial wavefunction $\psi(\boldsymbol{x},t_0)$ and the time-dependent potential V satisfy general conditions which ensure uniqueness and regularity for the solutions at later times, see for example Reed and Simon [1] (section X.12) and [2].

Here we investigate solutions of the TDSE with an exact time dependence for a range of models, firstly extending well-known results for the harmonic oscillator in one dimension to d dimensions, including the construction of dynamical invariant operators and their radial eigenfunctions such as time-dependent coherent states which satisfy the radial TDSE; this includes both Perelomov generalized coherent states and Barut–Girardello coherent states which appear in the context of $\mathfrak{su}(1,1)$ representations. Secondly, we extend these methods to other quantum-mechanical models in d dimensions showing how to construct the unitary time evolution operator, dynamical invariants and corresponding exact solutions including coherent state solutions. Thirdly, we show how to construct plane wave solutions and the subsequent general solution by means of the Fourier transform, and in specific models how then to construct distributional solutions of the TDSE for which the initial wavefunction ψ^0 is equal to the Dirac δ-function, from which the general solution may also be constructed. We briefly include known results in several sections, extensively referenced.

We consider Hamiltonians, including non-Hermitean Hamiltonians, which can be transformed by spatial unitary transformations comprising dilatations, translations and rotations, to an operator $F(H^0,t)$ which is a function of time and a time-independent operator H^0. Such Hamiltonians can also be constructed using dynamical invariants, which are quantum-mechanical constants of the motion and were used by Lewis and Riesenfeld [3] in 1969 to find exact solutions for the time-dependent harmonic oscillator. Invariants I, which need not be Hermitean, have the general property that for any solution ψ of the TDSE, $I|\psi\rangle$ is also a solution and so solutions can be constructed as eigenfunctions of I which has time-independent, possibly complex, eigenvalues. Alternatively, $I|\psi\rangle$ can be functionally independent of $|\psi\rangle$ and repeated application of I to $|\psi\rangle$ then generates a discrete set of solutions $I^n|\psi\rangle$ to the TDSE, in which I acts as a raising (creation) operator which is applied to a ground state $|\psi^0\rangle$, which in turn is found by solving $I^\dagger|\psi^0\rangle = 0$. We encounter this in the construction of Perelomov coherent states satisfying the TDSE in section 4.4, and also in the determination of ground states satisfying the TDSE.

Exact solutions of the TDSE have been extensively investigated by various means, including analysis of symmetries, often for particular Hamiltonians, see for example [5] where exact solutions for linear and quadratic time-dependent potentials are derived, also coordinate transformations [6, 7] possibly combined with separation of variables [8], see in particular [9] which obtains some results derived here by different methods, as well as unitary transformations and dynamical invariants. It follows from (1) that

$$\mathcal{H} = i\hbar\frac{\partial U}{\partial t}U^\dagger, \tag{2}$$

and hence under the transformation $U \to U' = T^\dagger U$, where T is unitary, we have $\mathcal{H} \to \mathcal{H}' = T^\dagger \mathcal{H} T - \mathrm{i}\hbar T^\dagger \partial T/\partial t$. We define $\mathcal{H}' = F(H^0, t)$, where H^0 is any time-independent operator which we subsequently diagonalize, and then construct $\mathcal{H}$ using T. Unitary transformations have been used previously to solve the TDSE, an example being the exact solutions found by Brown [10] in 1991 for the harmonic oscillator with periodic frequencies, although the solutions found there and also those found earlier by Husimi [11] in 1953 do not exist at all times due to the appearance of zeros in the denominator, as we discuss further in section 4, particularly section 4.6. Other work which uses unitary transformations to solve the TDSE is [12], related to our approach but restricted to dilatations for $d = 1$ and [13–18].

In section 2 we discuss general properties of dynamical invariants I and their construction by means of unitary transformations, and determine exact solutions of the TDSE as eigenfunctions of I and an explicit form for the time evolution operator U. These formulae remain valid for non-Hermitean Hamiltonians provided the non-Hermitean operator H^0 has real eigenvalues. Then in section 3 we investigate unitary dilatation transformations for general potentials, and the construction of a dynamical invariant, and discuss several examples and generalizations to models such as effective mass Hamiltonians.

In section 4 we investigate the harmonic oscillator in d dimensions, where we determine exact radial solutions of the TDSE such as $\mathfrak{su}(1, 1)$ coherent states, also plane wave and distributional solutions. Solutions are found by means of dilatations where the time-dependent parameter ρ satisfies a well-known second-order differential equation (25), often referred to as the Ermakov equation. We show that this equation is invariant under the ζ-transformation (28) which enables us to determine the general solution given any particular solution, and with which we obtain solutions ψ_ζ to the TDSE which depend on an arbitrary complex parameter ζ satisfying $|\zeta| < 1$. The ζ-transformation performs SU(1, 1) transformations on dynamical invariants. We demonstrate in section 4.4 that the eigenfunctions of the Lewis and Riesenfeld quadratic invariant are related by the ζ-transformation to Perelomov generalized coherent states. In section 4.5 we investigate the algebra of dynamical invariants, which are labeled by two real parameters, and show specifically how sums and products of these invariants are related to given dynamical invariants. In section 4.6 we investigate plane wave solutions, corresponding to a Hermitean invariant which is linear in momentum and position operators, and show that the general solution can be expressed as an integral transform by writing the initial function as a Fourier transform and, for radial solutions, as a Hankel transform. We extend these solutions to distributional (i.e., weak) solutions of the TDSE for which the initial wavefunction is equal to $\delta(x)$, from which the general solution may be constructed also as an integral transform.

In section 5 we discuss time-dependent translations in d dimensions and as an example apply the results to the time-dependent linear potential, obtaining solutions in several forms including plane wave and distributional solutions. We also find explicit coherent state solutions, and discuss $\mathfrak{su}(1, 1)$ coherent states which evolve according to the corresponding TDSE. In section 6 we consider time-dependent rotations for $d = 2, 3$ and show that such transformations can be used to solve the TDSE for time-dependent Hamiltonians describing electromagnetic interactions, with time-dependent magnetic fields that need not be uniform. As an example we find solutions to the TDSE for a stationary magnetic monopole in the presence of time-dependent magnetic and electric fields.

Momentum and position operators are denoted $\boldsymbol{p}$, $\boldsymbol{q}$, and act on states $|\psi\rangle$ in a Hilbert space of wavefunctions denoted $\psi(\boldsymbol{x}, t)$ according to the usual representation $\boldsymbol{p} \to -\mathrm{i}\hbar\nabla$, $\boldsymbol{q} \to \boldsymbol{x}$, where the inner product in d dimensions is given by $\langle\psi_2 \,|\, \psi_1\rangle = \int \overline{\psi}_2(\boldsymbol{x})\psi_1(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. For $d = 1$ we have $p \to -\mathrm{i}\hbar\,\mathrm{d}/\mathrm{d}x, q \to x$. Quadratic and linear combinations of position and momentum operators generate the inhomogeneous real symplectic group $\mathrm{Sp}(2d, \mathbb{R})$ (see for example [19]) and dilatations, translations, rotations are generated by corresponding subalgebras.

2. Dynamical invariants and unitary transformations

Solutions of the TDSE can be determined by first finding an explicit time-dependent operator I, a dynamical invariant, which commutes with $\mathrm{i}\hbar\frac{\partial}{\partial t} - \mathcal{H}(t)$ and which therefore satisfies

$$\mathrm{i}\hbar\frac{\partial I}{\partial t} - [\mathcal{H}, I] = 0. \tag{3}$$

Hence, for any solution ψ of the TDSE, $I|\psi\rangle$ is also a solution. Such an operator I may be regarded as a constant of the motion since the expectation value $\langle\psi|I|\psi\rangle$ is constant in time for any solution $|\psi\rangle$ of the TDSE. If I is Hermitean then its eigenvalues λ are real and constant in time [3] and given the orthonormal eigenfunctions $\phi^I_\lambda(\boldsymbol{x}, t)$, the functions

$$\psi_\lambda(\boldsymbol{x}, t) = \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)/\hbar}\phi^I_\lambda(\boldsymbol{x}, t) \tag{4}$$

solve the TDSE, where the time-dependent phase factor $\mathrm{e}^{\mathrm{i}\alpha_\lambda(t)/\hbar}$ can be determined by direct substitution into the TDSE.

There exist in principle many dynamical invariants for any given $\mathcal{H}$, corresponding to different initial wavefunctions ψ^0. There may exist a set of explicit invariants indexed by one or more parameters, as we discuss in section 4.5 for the harmonic oscillator, and there may also exist explicit functionally independent invariants. For any two invariants I_1, I_2 and constants c_1, c_2 the sum $c_1I_1 + c_2I_2$ and the product I_1I_2 are also invariants, and so the set of all invariants forms an algebra. Generally, given an explicit invariant I, any function $f(I)$ of I is also an invariant [61]. Similarly, to any given invariant I of $\mathcal{H}$ there are many Hamiltonians for which I is an invariant, for example I is also an invariant for the Hamiltonian $\mathcal{H}' = \mathcal{H} + F(I, t)$, where F is any real function of I and t. In this case the phase of the solution (4) of the TDSE with respect to $\mathcal{H}'$ changes according to $\alpha_\lambda \to \alpha_\lambda - \int F(\lambda, t)\,\mathrm{d}t$.

For any given $\mathcal{H}$ there is no general procedure to find an explicit invariant I, however if $\mathcal{H}$ is an element of a dynamical algebra $\mathcal{A}$ then we may expand I in terms of elements of $\mathcal{A}$, with time-dependent coefficients. Then (3) reduces to a set of ordinary differential equations for these coefficients, see for example [20] (section IVA), also Perelomov [21] (section 18.4). The time-dependent harmonic oscillator, for which the dynamical algebra is $\mathfrak{su}(1, 1)$, is one example of this approach, as followed by Lewis and Riesenfeld [3]. More directly, one can express the time evolution operator U as a general element of the group corresponding to $\mathcal{A}$, with time-dependent parameters, and then use (2) to solve for these parameters, see [22–31].

We proceed by constructing firstly a time-dependent operator I and then defining a Hamiltonian $\mathcal{H}$ for which I is an invariant. Let H^0 be a Hermitean time-independent operator, a function of $\boldsymbol{p}$ and $\boldsymbol{q}$, such as a time-independent Hamiltonian, and let T be any time-dependent unitary transformation, also a function of $\boldsymbol{p}, \boldsymbol{q}$. Define $I = TH^0T^\dagger$, and the time-dependent Hamiltonian

$$\mathcal{H} = \mathrm{i}\hbar\frac{\partial T}{\partial t}T^\dagger + F(I, t), \tag{5}$$

where $F(I, t) = TF(H^0, t)T^\dagger$ is any real function of I and t. $\mathcal{H}$ is Hermitean and I is an invariant for $\mathcal{H}$, as may be verified directly. We diagonalize H^0, i.e. $H^0\phi^0_\lambda = \lambda\phi^0_\lambda$ where λ is real, then I has time-independent eigenvalues λ and eigenfunctions $\phi^I_\lambda(\boldsymbol{x}, t) = T(t)\phi^0_\lambda(\boldsymbol{x})$.

The function $\psi_\lambda(\boldsymbol{x}, t)$ defined by (4) satisfies the TDSE provided that $\dot{\alpha}_\lambda = -F(\lambda, t)$, as is verified by direct substitution. Hence a set of solutions is given by

$$\psi_\lambda(\boldsymbol{x}, t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} F(\lambda, s)\,\mathrm{d}s\right] T(t)\phi_\lambda^0(\boldsymbol{x}), \tag{6}$$

where we have chosen $\alpha_\lambda(t_0) = 0$ at the initial time $t = t_0$, corresponding to the choice of initial wavefunction $\psi_\lambda^0(\boldsymbol{x}) = T(t_0)\phi_\lambda^0(\boldsymbol{x})$.

This construction can be generalized in various ways, for example if $\boldsymbol{H}^0 = \left(H_1^0, \ldots, H_n^0\right)$ denotes a time-independent vector operator with commuting components, then the components of the vector $\boldsymbol{I} = T\boldsymbol{H}^0 T^\dagger$ form a commuting set of invariants for the Hamiltonian

$$\mathcal{H} = \mathrm{i}\hbar\frac{\partial T}{\partial t}T^\dagger + F(\boldsymbol{I}, t),$$

where F is any real function of its arguments, and exact solutions ψ to the TDSE may then be expressed as a product of the eigenfunctions $\phi_{\lambda_j}^0(\boldsymbol{x})$ of each operator H_j^0, multiplied by a phase, namely:

$$\psi_{\lambda_1,\ldots,\lambda_n}(\boldsymbol{x}, t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} F(\lambda_1, \ldots, \lambda_n, s)\,\mathrm{d}s\right] T(t)\phi_{\lambda_1}^0(\boldsymbol{x})\cdots\phi_{\lambda_n}^0(\boldsymbol{x}). \tag{7}$$

Due to the unitary evolution of solutions of the TDSE we have, directly from (6), $\langle\psi_\lambda|\psi_{\lambda'}\rangle = \left\langle\phi_\lambda^0 \mid \phi_{\lambda'}^0\right\rangle$, and so the set of solutions $\{\psi_\lambda\}$ is orthonormal provided the set $\left\{\phi_\lambda^0\right\}$ is orthonormal. We may identify the unitary time evolution operator U explicitly from (6):

$$U(t, t_0) = U(t_0, t)^\dagger = T(t)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} F(H^0, s)\,\mathrm{d}s\right) T^\dagger(t_0), \tag{8}$$

and U satisfies the TDSE in the form (1) with $\mathcal{H}$ given by (5), with the initial condition $U(t_0, t_0) = \mathbb{I}$. U also satisfies $U(t, t_0) = U(t, t_1)U(t_1, t_0)$ for any intermediate time t_1. The time-dependent energy levels, i.e. the expectation values of $\mathcal{H}$ with respect to the states $|\psi_\lambda(t)\rangle$, are given by

$$\langle\psi_\lambda|\mathcal{H}|\psi_\lambda\rangle = F(\lambda, t) + \mathrm{i}\hbar\left\langle\phi_\lambda^0\right|T^\dagger\frac{\partial T}{\partial t}\left|\phi_\lambda^0\right\rangle, \tag{9}$$

as follows from (5) and (6) for $\mathcal{H}$ and ψ_λ.

Having suitably chosen H^0, T, F in order to construct a Hamiltonian in the form (5), we may then determine an invariant for any solution of the TDSE by using the fact that $U\mathcal{O}U^\dagger$ is an invariant for any operator $\mathcal{O}$, where U is given by (8). If the initial wavefunction $\psi^0 = \psi(\boldsymbol{x}, t_0)$ is an eigenfunction of, for example, the projection $\mathcal{O} = |\psi^0\rangle\langle\psi^0|$ then the solution $\psi = U\psi^0$ of the TDSE at time t is an eigenfunction of the invariant $I = U\mathcal{O}U^\dagger = |\psi\rangle\langle\psi|$. A useful choice for $d = 1$ is the invariant $I = UpU^\dagger$ where p is the momentum operator with eigenfunctions $\phi_k = \mathrm{e}^{\mathrm{i}kx}$, where U is given by (8), and $\psi_k = U\phi_k$ then solves the TDSE. By expressing any initial wavefunction ψ^0 as a Fourier transform, i.e. as a linear superposition of the plane waves ϕ_k, the general solution ψ of the TDSE can be expressed as an integral transform. We investigate this construction explicitly for the harmonic oscillator in section 4.6, also in section 5. For general d one can choose the vector invariant $\boldsymbol{I} = U\boldsymbol{p}U^\dagger$, with commuting components, and similarly express the initial wavefunction ψ^0 as a d-dimensional Fourier transform, with a corresponding solution $\psi = U\psi^0$. Another convenient choice of vector invariant is $\boldsymbol{a} = U\boldsymbol{a}^0U^\dagger$ where $\boldsymbol{a}^0$ denotes a d-dimensional time-independent vector of commuting boson annihilation operators, the eigenfunctions of which are coherent states, which we consider for the harmonic oscillator in section 4.3, and for the linear potential in section 5.

Conventional Hamiltonians $\mathcal{H}$ are quadratic in the momentum $\boldsymbol{p}$, possibly with linear terms, and since the operator $\frac{\partial T}{\partial t}T^\dagger$ is linear in $\boldsymbol{p}$ for the spatial unitary transformations T considered here, we generally choose H^0 to be quadratic or linear in $\boldsymbol{p}$, and $F(H^0, t)$ to be at most quadratic in H^0. Invariants which are at most quadratic in momentum have been discussed for general models by Ray [32] and by Efthimiou and Spector [9]. The construction we have outlined also applies to the non-Hermitean Hamiltonians extensively investigated in recent years, see the review by Bender [33]. In $\mathcal{PT}$-symmetric models the non-Hermitean time-independent Hamiltonian H^0 has a real, positive spectrum and time evolution is unitary in the sense that probability is conserved. For any unitary operator T the non-Hermitean invariant $I = TH^0T^\dagger$ therefore also has a real, positive spectrum. Although the Hamiltonian $\mathcal{H}$ defined by (5), where F is real, is no longer Hermitean since $F(I, t) \neq F(I^\dagger, t)$, nevertheless the functions (6) still exactly solve the TDSE and also preserve probability as time evolves, since λ is real. However the operator U defined by (8), which has the property that $\psi_\lambda(\boldsymbol{x}, t) = U(t, t_0)\psi_\lambda^0(\boldsymbol{x})$, and which is the unique solution of the TDSE with the initial condition $U(t_0, t_0) = \mathbb{I}$, is not unitary, consistent with the fact that $\mathcal{H}$ as expressed in (2) is no longer Hermitean.

3. Dilatations

We now investigate solutions of the TDSE in d dimensions in which the Hamiltonian $\mathcal{H}$ is constructed according to (5) where T includes a time-dependent dilatation. We choose

$$H^0 = \frac{\boldsymbol{p}^2}{2m} + U(\boldsymbol{q}) \tag{10}$$

for some potential U, and denote by $\phi_\lambda^0(\boldsymbol{x})$ the eigenfunctions corresponding to the eigenvalue λ. We also choose $F(I, t) = I/\sigma(t)^2$, where σ is a strictly positive function of t for $t \geqslant t_0$, and where $I = TH^0T^\dagger$ in which $T = T_{\mathrm{q}}T_{\mathrm{dil}}$ is the product of two spatial unitary transformations, also elements of the group SU(1, 1). Define

$$T_{\mathrm{dil}}[\rho] = \exp\left[-\frac{\mathrm{i}\log\rho}{2\hbar}(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p})\right] \tag{11}$$

for any positive function ρ, which performs dilatations, specifically:

$$T_{\mathrm{dil}}f(\boldsymbol{q})T_{\mathrm{dil}}^\dagger = f\left(\frac{\boldsymbol{q}}{\rho}\right), \qquad T_{\mathrm{dil}}f(\boldsymbol{p})T_{\mathrm{dil}}^\dagger = f(\rho\boldsymbol{p}), \tag{12}$$

for general functions f. Let

$$T_{\mathrm{q}}[\alpha] = \exp\left[\frac{\mathrm{i}\alpha}{2\hbar}\boldsymbol{q}^2\right], \tag{13}$$

where $\alpha = \alpha(t)$ is any function of t, then $T_{\mathrm{q}}\boldsymbol{p}T_{\mathrm{q}}^\dagger = \boldsymbol{p} - \alpha\boldsymbol{q}$ and so

$$T_{\mathrm{q}}\boldsymbol{p}^2T_{\mathrm{q}}^\dagger = \boldsymbol{p}^2 + \alpha^2\boldsymbol{q}^2 - \alpha(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}).$$

We obtain therefore the invariant

$$I = TH^0T^\dagger = \frac{\rho^2}{2m}\boldsymbol{p}^2 + \frac{\alpha^2\rho^2}{2m}\boldsymbol{q}^2 - \frac{\alpha\rho^2}{2m}(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}) + U\left(\frac{\boldsymbol{q}}{\rho}\right). \tag{14}$$

We also find

$$\mathrm{i}\hbar\frac{\partial T}{\partial t}T^\dagger = -\left(\frac{\alpha\dot\rho}{\rho} + \frac{\dot\alpha}{2}\right)\boldsymbol{q}^2 + \frac{\dot\rho}{2\rho}(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}),$$

then the time-dependent Hamiltonian defined by (5) is given by

$$\mathcal{H} = \frac{\rho^2}{2m\sigma^2}\boldsymbol{p}^2 + \left(\frac{\alpha^2\rho^2}{2m\sigma^2} - \frac{\alpha\dot{\rho}}{\rho} - \frac{\dot{\alpha}}{2}\right)\boldsymbol{q}^2 + \left(\frac{\dot{\rho}}{2\rho} - \frac{\alpha\rho^2}{2m\sigma^2}\right)(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}) + \frac{1}{\sigma^2}U\left(\frac{\boldsymbol{q}}{\rho}\right), \tag{15}$$

and the corresponding TDSE has the exact solutions (6), specifically:

$$\psi_\lambda(\boldsymbol{x}, t) = \rho^{-\frac{d}{2}} \exp\left(-\frac{\mathrm{i}\lambda}{\hbar}\int_{t_0}^{t}\frac{\mathrm{d}s}{\sigma(s)^2}\right)\exp\left(\frac{\mathrm{i}\alpha\boldsymbol{x}^2}{2\hbar}\right)\phi_\lambda^0\left(\frac{\boldsymbol{x}}{\rho}\right), \tag{16}$$

with the corresponding time-evolution operator given by (8). The energy levels with respect to the solutions (16) are given by (9), explicitly:

$$\langle\psi_\lambda|\mathcal{H}|\psi_\lambda\rangle = \frac{\lambda}{\sigma^2} - \frac{\dot{\alpha}\rho^2}{2}\langle\phi_\lambda^0|\boldsymbol{q}^2|\phi_\lambda^0\rangle + \frac{\dot{\rho}}{2\rho}\langle\phi_\lambda^0|(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p})|\phi_\lambda^0\rangle. \tag{17}$$

The Hamiltonian (15) describes a system with a time-dependent mass $M(t) = m\sigma^2/\rho^2$. We may choose the functions ρ, σ, α to match any given time-dependent coefficients of $\boldsymbol{p}^2$, $\boldsymbol{q}^2$ and $\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}$, in particular for $\sigma = \rho$ and $\alpha = m\dot{\rho}/\rho$ we obtain solutions for the Hamiltonian $\mathcal{H} = \boldsymbol{p}^2/2m + V(\boldsymbol{q}, t)$ where

$$V(\boldsymbol{q}, t) = \frac{1}{\rho^2}U\left(\frac{\boldsymbol{q}}{\rho}\right) - \frac{m\ddot{\rho}\boldsymbol{q}^2}{2\rho}. \tag{18}$$

The solutions (16) for this particular case have been previously derived by Hartley and Ray [6, 34] (by means of invariants) and also in [12] (using unitary transformations) for $d = 1$, and for general d in [9] (equation (4.20) including time-dependent translations) by means of coordinate transformations.

3.1. Examples and generalizations

The Hamiltonian (15) with $\sigma = \rho$ and $\alpha = m\dot{\rho}/\rho$ is of the form $\mathcal{H} = \boldsymbol{p}^2/2m + V(\boldsymbol{x}, t)$, where V is given in terms of U by (18). If U is radial then V is also radial, in particular if U is quadratic in $r = |\boldsymbol{x}|$, then so is V, which allows one to find solutions of the TDSE for the radial harmonic oscillator for general time-dependent frequencies, as we discuss in detail in section 4. If U and hence V are not radial, one can still find exact solutions to the TDSE for models in which the time-independent problem can be solved exactly; some examples of non-central potentials are discussed in [35, 36]. If V is radial but does not contain any terms quadratic in r then we must have

$$\rho(t)^2 = \frac{\omega_0^2}{a^2} + (at + b)^2, \tag{19}$$

where $a > 0$ and b, ω_0 are constants, as applies to models such as the hydrogen atom for which $V(r, t) = Z(t)/r$ where the time-dependent charge is given by $Z(t) = Z_0/\rho(t)$ for constant Z_0. The case $\omega_0 = 0$ for which $\rho(t) = at + b$ has been discussed in various papers [37–43].

An example widely investigated is the infinite quantum well with moving walls, generally in $d = 1$ dimensions [37, 38, 40–49]. For the radial case we choose $U(r) = 0$ for $r \leqslant R$ and the radial (unnormalized) eigenfunctions of H^0 are proportional to Bessel functions:

$$\phi_\kappa^0(r) = r^{1-\frac{d}{2}} J_{\ell-1+\frac{d}{2}}(\kappa r),$$

where the angular momentum ℓ takes integer values, κ is related to the eigenvalue λ of H^0 by $\lambda = \kappa^2\hbar^2/(2m)$ and κ is determined by the requirement that $\phi_\kappa^0(r)$ vanish at the boundary

$r = R$. The time-dependent potential $V(r,t)$ is given by (18), that is, $V(r,t) = -m\ddot{\rho}r^2/(2\rho)$ for $r \leqslant R\rho$, and infinity otherwise, where ρ is the positive dilatation function. Radial solutions $\psi_\kappa(r,t)$ of the TDSE are given by (16), and are confined by the moving boundary. The infinite well for $d = 1$ can be generalized by means of time-dependent translations (section 5) to include independently moving left and right walls, and one may also combine dilatations and translations in higher dimensions in order to describe infinite quantum wells with rectangular moving walls.

The formalism outlined in section 2 applies also to non-Hermitean Hamiltonians, for example if for $d = 1$ we choose $U(x) = x^2(\mathrm{i}x)^\epsilon$ where $\epsilon \geqslant 0$ then $H^0 = p^2/2m + U(x)$ has real eigenvalues [33], and the Hamiltonian $\mathcal{H}$ with V given by (18), namely

$$V(x,t) = \frac{x^2}{\rho^{4+\epsilon}}(\mathrm{i}x)^\epsilon + \frac{m\ddot{\rho}}{2\rho}x^2,$$

allows solutions of the TDSE as in (16).

The solution (16) to the TDSE with the potential V given by (18) can be generalized in many ways, two examples of which we discuss. In the first we consider dilatations which are applied independently to each coordinate, and in the second we investigate solutions of the TDSE for a position-dependent mass, in this case for $d = 1$ only.

Let H^0 be as in (10) with eigenfunctions $\phi^0_\lambda(\boldsymbol{x})$ corresponding to eigenvalues λ, and let $\rho_1, \ldots, \rho_d$ be a set of d functions of t which are each strictly positive for $t \geqslant t_0$, and define the transformed vector

$$\boldsymbol{x}_\rho = \left(\frac{x_1}{\rho_1}, \ldots, \frac{x_d}{\rho_d}\right),$$

and the potential

$$V(\boldsymbol{x},t) = \frac{1}{\rho^2}U(\boldsymbol{x}_\rho) + m\sum_{j=1}^d x_j^2\left(\frac{\rho^2\dot{\rho}_j^2}{\rho_j^4} - \frac{\rho\dot{\rho}\dot{\rho}_j}{\rho_j^3} - \frac{\rho^2\ddot{\rho}_j}{2\rho_j^3}\right),$$

where ρ, also a positive function of t, is introduced for convenience. The time-dependent Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m\rho^2}\sum_{j=1}^d \rho_j^2\frac{\partial^2}{\partial x_j^2} + V(\boldsymbol{x},t)$$

describes a system with a non-isotropic time-dependent mass $m_j(t) = m\rho^2\big/\rho_j^2$. Solutions to the TDSE are given by

$$\psi_\lambda(\boldsymbol{x},t) = (\rho_1\rho_2\cdots\rho_d)^{-\frac{1}{2}}\exp\left(-\frac{\mathrm{i}\lambda\tau}{\hbar}\right)\exp\left(\frac{\mathrm{i}m\rho^2}{2\hbar}\sum_{j=1}^d\frac{x_j^2\dot{\rho}_j}{\rho_j^3}\right)\phi^0_\lambda(\boldsymbol{x}_\rho), \tag{20}$$

where τ is given by

$$\tau(t) = \int_{t_0}^t \frac{\mathrm{d}s}{\rho(s)^2}, \tag{21}$$

and reduces to (16) when $\rho_j = \rho$ for each $j = 1, \ldots, d$.

This solution corresponds to the choice of dynamical invariant $I = TH^0T^\dagger$ where $T = T_1T_2\cdots T_d$ is a product of commuting factors T_j with

$$T_j = \exp\left[\frac{\mathrm{i}m\rho^2\dot{\rho}_jq_j^2}{2\hbar\rho_j^3}\right]\exp\left[-\frac{\mathrm{i}\log\rho_j}{2\hbar}(p_jq_j + q_jp_j)\right]$$

for $j = 1, \ldots, d$, where the second factor performs a dilatation on x_j, and I is given explicitly by

$$I = \frac{1}{2m} \sum_{j=1}^{d} \rho_j^2 (p_j - \alpha_j q_j)^2 + U(\boldsymbol{x}_\rho),$$

where $\alpha_j = m\rho^2 \dot{\rho}_j / \rho_j^3$, and the wavefunction ψ_λ in (20) is an eigenfunction of I with eigenvalue λ. The Hamiltonian is given by (5) with $F(I,t) = I/\rho(t)^2$, and the solution to the TDSE follows then from (6).

As a second generalization, for $d = 1$ only, consider the effective mass Hamiltonian

$$H^0 = p \frac{1}{2m(q)} p + U(q),$$

where the mass $m = m(q)$ is a positive function of q and U is a potential function. We assume that the non-commuting operators in H^0 can be ordered into the form shown. Such Hamiltonians, with a position-dependent effective mass, have been used to model the motion of electrons and holes in semiconductors [50] and quantum liquids [51], and exact solutions for the time-independent case have recently been investigated by various authors, see for example [52–59]. As before, denote by $\phi_\lambda^0(x)$ the eigenfunction of H^0 corresponding to the eigenvalue λ, and define $\mu(x) = \int_0^x sm(s)\,\mathrm{d}s$. Let G be any positive function of the strictly positive dilatation function ρ, and let $\mathcal{H}(t)$ be the time-dependent Hamiltonian

$$\mathcal{H} = -\hbar^2 \frac{\partial}{\partial x} \frac{1}{2M(x,t)} \frac{\partial}{\partial x} + V(x,t),$$

where the time- and position-dependent mass M is given by $M(x,t) = G(\rho)m(x_\rho)/\rho^2$, where $x_\rho = x/\rho$, and the potential is given by

$$V(x,t) = \frac{U(x_\rho)}{G(\rho)} + \frac{G(\rho)\dot{\rho}^2}{2\rho^4} x^2 m(x_\rho) + \left[\frac{G(\rho)\dot{\rho}^2}{\rho^2} - \frac{G\prime(\rho)\dot{\rho}^2}{\rho} - \frac{G(\rho)\ddot{\rho}}{\rho} \right] \mu(x_\rho).$$

An exact solution of the TDSE is

$$\psi_\lambda(x,t) = \rho^{-\frac{1}{2}} \exp\left[-\frac{\mathrm{i}\lambda}{\hbar} \int_{t_0}^{t} \frac{\mathrm{d}s}{G(\rho(s))} \right] \exp\left[\frac{\mathrm{i}G(\rho)\dot{\rho}}{\hbar\rho} \mu(x_\rho) \right] \phi_\lambda^0(x_\rho). \qquad (22)$$

In the notation of section 2 the transformation T is given by

$$T = \exp\left[-\frac{\mathrm{i}\log\rho}{2\hbar}(pq + qp) \right] \exp\left[\frac{\mathrm{i}G(\rho)\dot{\rho}}{\hbar\rho} \mu(q) \right],$$

and the invariant I is

$$I = \rho^2 p \frac{1}{2m(q_\rho)} p - \frac{G(\rho)\dot{\rho}}{2\rho}(pq + qp) + \frac{G(\rho)^2\dot{\rho}^2}{2\rho^4} m(q_\rho) q^2 + U(q_\rho),$$

where $q_\rho = q/\rho$. The Hamiltonian $\mathcal{H}$ is constructed according to (5) with $F(I,t) = I/G(\rho(t))$. The solution (22) reduces to (16) for $d = 1$ if $m(x) = m$ is position independent and $G(\rho) = \rho^2$.

Various mass functions have been considered in the literature, see for example [53], in particular if $m(x) = m_0|x|^\gamma$ where $\gamma > -2$ (as in [55], see example 5 in the Appendix), then for $G(\rho) = \rho^{\gamma+2}$ we find $M(x,t) = m(x)$ is independent of t.

4. The time-dependent harmonic oscillator

The time-dependent isotropic harmonic oscillator in d dimensions is defined by the Hamiltonian

$$\mathcal{H}(t) = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2} m\omega^2(t)\boldsymbol{q}^2, \tag{23}$$

where $\omega(t)$ is any given function of t. We find solutions of the TDSE by choosing $H^0 = H^0_{\mathrm{HO}}$ where

$$H^0_{\mathrm{HO}} = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2} m\omega_0^2\boldsymbol{q}^2 \tag{24}$$

($\omega_0 > 0$) with the corresponding potential V given by (18), which we consider for simplicity instead of the more general Hamiltonian $\mathcal{H}$ (15). The harmonic oscillator and this generalization have been widely investigated, see for example the selection of references [14, 26, 60–69].

4.1. The Ermakov equation and SU(1, 1) transformations

By comparing the potential in (23) with V in (18) we find that ρ is determined for any given $\omega(t)$ by the nonlinear equation

$$\ddot{\rho} + \omega(t)^2\rho = \frac{\omega_0^2}{\rho^3}, \tag{25}$$

often known as the Ermakov equation, solutions of which were investigated by Ermakov [70] in 1880 and later by Milne [71], Pinney [72], Lewis [73] and others. It is convenient in the following to retain explicitly the constant ω_0, which for $\omega_0 > 0$ can be set to unity by rescaling ρ, in order to investigate the limiting case $\omega_0 \to 0$ and also the possibility that $\omega_0^2 < 0$.

The general solution of (25) can be constructed [70–72] from solutions $f(t)$ of the linear equation of motion for the classical time-dependent harmonic oscillator,

$$\ddot{f} + \omega(t)^2 f = 0, \tag{26}$$

according to $\rho^2/\omega_0 = f_1^2 + W^{-2} f_2^2$, where f_1, f_2 are linearly independent solutions of (26), and where the Wronskian $W[f_1, f_2] = f_1\dot{f}_2 - \dot{f}_1 f_2$ is a nonzero constant. For example, if we choose a constant frequency $\omega(t) = \omega_0$ the general two-parameter solution (with $t_0 = 0$) is

$$\rho(t)^2 = \rho_0^2\cos^2\omega_0 t + \left(\frac{\rho_1^2}{\omega_0^2} + \frac{1}{\rho_0^2}\right)\sin^2\omega_0 t + \frac{\rho_0\rho_1}{\omega_0}\sin 2\omega_0 t, \tag{27}$$

where $\rho_0 = \rho(t_0) > 0$ and $\rho_1 = \dot{\rho}(t_0)$.

Conversely we may construct solutions of the linear equation (26) from any given solution ρ of the Ermakov equation (25). As observed in [70, 71], the linearly independent functions $f_1 = \rho\cos\omega_0\tau$ and $f_2 = \rho\sin\omega_0\tau$, where τ is defined by (21), each satisfy (26) and so the general complex solution of (26) is therefore a complex multiple of $(1 + \zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})\rho\,\mathrm{e}^{\mathrm{i}\omega_0\tau}$, where ζ is complex.

A useful method of constructing solutions of (25) is by means of the following ζ-transformation:

$$\rho \xrightarrow{\zeta} \rho_\zeta = \rho\left[\frac{(1 + \zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})(1 + \bar{\zeta}\,\mathrm{e}^{2\mathrm{i}\omega_0\tau})}{1 - |\zeta|^2}\right]^{\frac{1}{2}}, \tag{28}$$

where ζ is any complex number with $|\zeta| < 1$, which leaves the Ermakov equation invariant, i.e. if ρ satisfies (25) then so does ρ_ζ. Hence, given any special solution ρ of the Ermakov equation we may construct the general solution ρ_ζ, depending on two real constants, by using the transformation (28). For example, if ρ is the unique solution satisfying $\rho_0 = 1, \rho_1 = 0$ then the general solution ρ_ζ for any given initial conditions $\rho_\zeta(t_0) > 0$ and $\dot\rho_\zeta(t_0)$ is given by (28) where ζ can be chosen to satisfy

$$\rho_\zeta(t_0) = \frac{|1+\zeta|}{\sqrt{1-|\zeta|^2}}, \qquad \dot\rho_\zeta(t_0) = \frac{2\omega_0 \Im(\zeta)}{|1+\zeta|\sqrt{1-|\zeta|^2}}.$$

If $\omega(t) = \omega_0$ then (28) transforms the specific solution $\rho(t) = 1$ to the general solution (27).

Two successive transformations (28) with corresponding parameters ζ_1, ζ_2, that is $\rho \xrightarrow{\zeta_1} \rho_{\zeta_1} \xrightarrow{\zeta_2} \rho_\zeta$, are equivalent to a single transformation $\rho \xrightarrow{\zeta} \rho_\zeta$ where

$$\zeta = \frac{\zeta_1(1+\overline{\zeta_1}) + \zeta_2(1+\zeta_1)}{1+\overline{\zeta_1}+\overline{\zeta_1}\zeta_2(1+\zeta_1)}, \tag{29}$$

as follows from

$$\mathrm{e}^{2\mathrm{i}\omega_0\tau_\zeta} = \frac{\mathrm{e}^{2\mathrm{i}\omega_0\tau}(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})(1+\overline{\zeta})}{(1+\overline{\zeta}\,\mathrm{e}^{2\mathrm{i}\omega_0\tau})(1+\zeta)}, \tag{30}$$

where

$$\tau_\zeta(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{\rho_\zeta(s)^2}. \tag{31}$$

The ζ-transformation corresponds to an element of the group SU(1, 1), specifically, let

$$g_\zeta = \frac{1}{\sqrt{1-|\zeta|^2}} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\delta} & -\zeta\,\mathrm{e}^{-\mathrm{i}\delta} \\ -\overline{\zeta}\,\mathrm{e}^{\mathrm{i}\delta} & \mathrm{e}^{\mathrm{i}\delta} \end{pmatrix}, \tag{32}$$

where the phase δ is defined by

$$\mathrm{e}^{\mathrm{i}\delta} = \frac{1+\zeta}{|1+\zeta|}, \tag{33}$$

then g_ζ is an element of SU(1, 1) and the group composition law $g_{\zeta_2}g_{\zeta_1} = g_\zeta$ holds with ζ given in terms of ζ_1, ζ_2 by (29). The transformation $\rho_\zeta \xrightarrow{\zeta'} \rho$ inverse to (28) is therefore determined by the parameter $\zeta' = -\zeta(1+\overline{\zeta})/(1+\zeta)$. Equations (28) and (30) follow from

$$\rho_\zeta\,\mathrm{e}^{\mathrm{i}\omega_0\tau_\zeta} = \frac{1+\overline{\zeta}}{|1+\zeta|\sqrt{1-|\zeta|^2}}(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})\rho\,\mathrm{e}^{\mathrm{i}\omega_0\tau}, \tag{34}$$

where each side is a general complex solution of (26), and where the multiplicative constant on the right is determined in magnitude by the requirement that ρ_ζ satisfy (25).

We encounter the ζ-transformation (28) specifically in section 4.4 as a group operation in relation to Perelomov coherent states which are defined using $\mathfrak{su}(1,1)$ properties, and which depend explicitly on ζ. We show that (28) performs an SU(1, 1) transformation on invariant boson operators $a, a^\dagger$ to form invariants $a_\zeta, a_\zeta^\dagger$, where the matrix transformation is performed by g_ζ as defined in (32) (see (68)).

A fundamental property of the Ermakov equation is that the solution ρ is nonzero for all t (see appendix B in [74], and the discussion in appendix B of [75]), and hence for $\rho_0 > 0$ we have $\rho(t) > 0$ for all t. The dilatation operator (11) and the subsequent transformed functions are therefore well defined for all t. By contrast real solutions of (26), corresponding to the choice $\omega_0 = 0$ in (25), generally have an infinite number of zeros at which times the dilatation transformation is undefined. It is known (see for example [111], chapter 11) that if $\omega(t)$ is continuous for $t \geqslant t_0$ and there exists $\kappa > 0$ such that

$$\omega(t) \geqslant \kappa > 0 \qquad \text{for all} \quad t \geqslant t_0, \tag{35}$$

then any real solution of $\ddot{f} + \omega(t)^2 f = 0$ has infinitely many zeros for $t \geqslant t_0$. Solutions of the TDSE based on a dilatation function ρ satisfying (25) with $\omega_0 = 0$ therefore exist generally only locally in the small time interval $t \geqslant t_0$ for which $\rho > 0$, including those found by Brown [10], also [76] and others [77, 79, 11]. If the condition (35) is not satisfied then solutions f which are positive for all $t \geqslant t_0$ are possible, for example if $\omega(t) = \omega_0/(1 + 2\omega_0 t)$ where $\omega_0 > 0$, then a solution which is positive for all $t > 0$ is $f(t) = \sqrt{1 + 2\omega_0 t}$.

4.2. The quadratic Hermitean invariant

The Hermitean invariant (14) is given by

$$I[\rho] = \frac{\rho^2}{2m}\boldsymbol{p}^2 + \frac{m}{2}\left(\dot{\rho}^2 + \frac{\omega_0^2}{\rho^2}\right)\boldsymbol{q}^2 - \frac{\rho\dot{\rho}}{2}(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}), \tag{36}$$

and has eigenvalues λ_n given by

$$\lambda_n = \left(2n + \ell + \frac{d}{2}\right)\hbar\omega_0, \tag{37}$$

with $n = 0, 1, \ldots$. The eigenfunctions $\phi_\lambda^0(\boldsymbol{x})$ of H^0 are found by separating variables in the usual way as a product of angular and radial functions, where the angular components are time independent and depend on the angular momentum $\ell = 0, 1, \ldots$. The radial solutions can be expressed in terms of generalized Laguerre polynomials, see for example Stillinger [81], and appear as a consequence of the $\mathfrak{su}(1,1)$ symmetry of the harmonic oscillator. Properties of Laguerre polynomials and their connection with $\mathfrak{su}(1,1)$ have been discussed and summarized by Biedenharn and Louck [82] (see Topic 6, pp 284, 304). The corresponding solution $\psi_\ell(\boldsymbol{x}, t)$ of the TDSE, with definite angular momentum ℓ, also separates into a product of time-independent angular functions and time-dependent radial functions $\psi_{n,\ell}(r, t)$. Hence, from (6) we find that the radial solution of the TDSE is

$$\begin{aligned}\psi_{n,\ell}(r,t) = (-1)^n &\sqrt{\frac{n!}{\Gamma\left(n + \ell + \frac{d}{2}\right)}}\left(\frac{m\omega_0}{\hbar\rho^2}\right)^{\frac{1}{4}(d+2\ell)} r^\ell L_n^{(\ell-1+\frac{d}{2})}\left(\frac{m\omega_0 r^2}{\hbar\rho^2}\right)\\ &\times \exp\left(-\frac{m\omega_0 r^2}{2\hbar\rho^2}\right)\exp\left(\frac{\mathrm{i}m\dot{\rho}r^2}{2\hbar\rho}\right)\exp\left[-\frac{\mathrm{i}(4n + 2\ell + d)\tau}{2\hbar}\right],\end{aligned} \tag{38}$$

where L denotes generalized Laguerre polynomials, and where τ is defined in (21). The solution depends on the initial values $\rho_0 = \rho(t_0)$ and $\rho_1 = \dot{\rho}(t_0)$ but is independent of ω_0, which can be rescaled to unity. The dependence of the solution $\psi_{n,\ell}$ on two real parameters related to ρ_0, ρ_1 becomes apparent by means of the ζ-transformation (28) in which we replace $\rho \to \rho_\zeta$ as discussed further in section 4.4.

The solutions $\psi_{n,\ell}$ are normalized at equal times with respect to the inner product

$$\langle\psi_1|\psi_2\rangle = 2\int_0^\infty r^{d-1}\overline{\psi_1(r,t)}\psi_2(r,t)\,\mathrm{d}r, \tag{39}$$

and for fixed ℓ are therefore orthonormal: $\langle\psi_{n,\ell}|\psi_{n',\ell}\rangle = \delta_{nn'}$. The explicit form (8) for the unitary time evolution operator, which is an element of SU(1, 1), reduces to

$$U(t,t_0) = T(t)\exp\left[-\frac{\mathrm{i}}{\hbar}\tau(t)\left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega_0^2\boldsymbol{q}^2\right)\right]T^\dagger(t_0), \tag{40}$$

where $T = T_q T_{dil}$ and the unitary transformations T_{dil}, T_q are given by (11) and (13). For $d = 1$ the Laguerre polynomials in (38) reduce to Hermite polynomials (see [83], p 779) and so the solutions (38) reduce to the well-known form

$$\psi_n(x,t) = \left(\frac{m\omega_0}{2^{2n}n!^2\pi\hbar\rho^2}\right)^{\frac{1}{4}} H_n\left(\frac{x}{\rho}\sqrt{\frac{m\omega_0}{\hbar}}\right)\exp\left(-\frac{m\omega_0 x^2}{2\hbar\rho^2}\right)\exp\left(\frac{\mathrm{i}m\dot{\rho}x^2}{2\hbar\rho}\right)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\omega_0\tau}, \tag{41}$$

where H denotes Hermite polynomials.

The energy levels $\langle\mathcal{H}\rangle$ with respect to the states $\psi_{n,\ell}$ may be calculated by means of the formula (17) with $\sigma = \rho$ and $\alpha = m\dot{\rho}/\rho$, or directly by determining

$$\langle \boldsymbol{p}^2\rangle = \frac{m\hbar}{2\omega_0}\left(\frac{\omega_0^2}{\rho^2}+\dot{\rho}^2\right)(4n+d+2\ell), \qquad \langle \boldsymbol{q}^2\rangle = \frac{\hbar\rho^2}{2m\omega_0}(4n+d+2\ell), \tag{42}$$

which leads to

$$\langle\psi_{n,\ell}|\mathcal{H}|\psi_{n,\ell}\rangle = \frac{\hbar}{4\omega_0}(4n+d+2\ell)\left(\frac{\omega_0^2}{\rho^2}+\dot{\rho}^2+\omega^2\rho^2\right). \tag{43}$$

Evidently the spacing of the levels is independent of n, as was observed for $d = 1$ by Lewis [73].

4.3. Non-Hermitean dynamical invariants and time-dependent coherent states for $d = 1$

Let $\mathcal{O}$ be any operator then $I = U\mathcal{O}U^\dagger$ is a dynamical invariant, where U is given in (40). We now choose $\mathcal{O}$ to be the time-independent annihilation operator a_0, the eigenfunctions of which are coherent states $\phi_\alpha^0(x)$ and determine time-dependent coherent states $\psi_\alpha(x,t) = U(t,t_0)\phi_\alpha^0(x)$ which satisfy the TDSE. Such states have been investigated, in particular by Malkin *et al* [4] in 1970, as eigenfunctions of boson operators which are dynamical invariants linear in position and momentum, and act as raising and lowering operators with respect to the solutions (41) for $d = 1$. We derive these coherent states directly using $U(t,t_0)$ as given in (40), where for convenience we choose $\rho_0 = \rho(t_0) = 1$, $\rho_1 = \dot{\rho}(t_0) = 0$, in which case $T(t_0) = \mathbb{I}$.

Let a_0 be the annihilation operator for the time-independent harmonic oscillator (with $\omega(t) = \omega_0$):

$$a_0 = \sqrt{\frac{m\omega_0}{2\hbar}}q + \frac{\mathrm{i}}{\sqrt{2m\omega_0\hbar}}p, \tag{44}$$

and define the boson operators $a[\rho] = Ua_0U^\dagger$, $a^\dagger[\rho] = Ua_0^\dagger U^\dagger$, which are functionals of ρ. Explicitly

$$a[\rho] = \frac{\mathrm{e}^{\mathrm{i}\omega_0\tau}}{\sqrt{2\hbar}}\left[\left(\frac{\sqrt{m\omega_0}}{\rho} - \frac{\mathrm{i}m\dot{\rho}}{\sqrt{m\omega_0}}\right)q + \frac{\mathrm{i}\rho}{\sqrt{m\omega_0}}p\right], \tag{45}$$

with $a^\dagger[\rho] = (a[\rho])^\dagger$, where τ is defined by (21). The creation and annihilation operators are related formally by $a^\dagger[\rho] = \mathrm{i}a[\mathrm{i}\rho]$. These operators differ from those defined by Lewis [73] by the inclusion of a time-dependent phase, but correspond to operators defined in [4, 26] where $\epsilon = \rho\,\mathrm{e}^{\mathrm{i}\omega_0\tau}$ is a complex solution to the classical equation (26), and are similar to those defined by Brown [10], except that here the function ρ is always strictly positive and hence τ and $a, a^\dagger$ are well defined at all times, unlike the corresponding operators in [10]. Other papers [84, 85] define similar creation and annihilation operators. The fact that $a, a^\dagger$ are invariants is well known [4, 63, 67, 95].

If we define the time-dependent vacuum state $|0, t\rangle$ in the usual way, according to $a|0, t\rangle = 0$, and the normalized states by $|n, t\rangle = (a^\dagger)^n|0, t\rangle/\sqrt{n!}$, then the solutions (41) of the TDSE are these same states in the coordinate representation: $\psi_n(x, t) = \langle x|n, t\rangle$, and $a^\dagger$, a act as raising and lowering operators with respect to the solutions (41), specifically we have $a\psi_n = \sqrt{n}\psi_{n-1}$, $a^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$. The invariant I given by (36) (for $d = 1$) is related to the boson number operator by $2I = \hbar\omega_0\{a, a^\dagger\}$.

Time-dependent coherent states, first discussed by Husimi [11], have been widely studied in various contexts, see particularly [4, 26], but also [60, 79, 86–94]. These states $|\alpha, t\rangle$, where α is complex, are eigenfunctions of the annihilation operator:

$$a|\alpha, t\rangle = \alpha|\alpha, t\rangle \tag{46}$$

where a is given by (45). Time-independent coherent states $|\alpha\rangle_0$ satisfy $a_0|\alpha\rangle_0 = \alpha|\alpha\rangle_0$ where the functions $\phi^0_\alpha(x) = \langle x|\alpha\rangle_0$ are given by

$$\phi^0_\alpha(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{-\alpha^2/2} \exp\left(-\frac{m\omega_0 x^2}{2\hbar} + \alpha x\sqrt{\frac{2m\omega_0}{\hbar}}\right). \tag{47}$$

We calculate the time-dependent coherent states $\psi_\alpha(x, t) = U(t, t_0)\phi^0_\alpha(x)$ by first writing U in the form $U = T_I T_{\mathrm{q}} T_{\mathrm{dil}}$ where $T_I = \exp(-\mathrm{i}\tau I/\hbar)$ and I is the invariant (36). Since $2I = \hbar\omega_0\{a, a^\dagger\}$ we have $T_I a T_I^\dagger = a\,\mathrm{e}^{\mathrm{i}\omega_0\tau}$ and so T_I multiplies α by a phase: $T_I|\alpha, t\rangle = |\alpha\,\mathrm{e}^{-\mathrm{i}\omega_0\tau}, t\rangle$. We obtain therefore the following time-dependent coherent states which satisfy the TDSE:

$$\psi_\alpha(x, t) = \left(\frac{m\omega_0}{\pi\hbar\rho^2}\right)^{\frac{1}{4}} \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{-\mathrm{i}\omega_0\tau/2} \exp\left(\frac{\mathrm{i}m\dot\rho x^2}{2\hbar\rho}\right) \exp\left(-\frac{\alpha^2\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau}}{2}\right)$$
$$\times \exp\left(-\frac{m\omega_0 x^2}{2\hbar\rho^2} + \frac{\alpha\,\mathrm{e}^{-\mathrm{i}\omega_0\tau}x}{\rho}\sqrt{\frac{2m\omega_0}{\hbar}}\right). \tag{48}$$

The function ρ is any particular solution of (25) but by means of the ζ-transformation (28) can be replaced by the general solution ρ_ζ depending on a complex parameter ζ. In this way, the coherent states ψ_α can be generalized to solutions $\psi_{\alpha,\zeta}$ depending on two complex parameters α, ζ.

Another method of derivation, which we follow for $\mathfrak{su}(1, 1)$ coherent states, proceeds as for the time-independent case [96]:

$$\psi_\alpha(x, t) = \langle x|\,\mathrm{e}^{\alpha a^\dagger - \bar\alpha a}|0, t\rangle = \mathrm{e}^{-|\alpha|^2/2}\langle x|\,\mathrm{e}^{\alpha a^\dagger}|0, t\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle x|n, t\rangle, \tag{49}$$

where the functions $\langle x|n, t\rangle = \psi_n(x, t)$ are given in (41). By means of the generating function for Hermite polynomials (see [83], p 784) we obtain the states (48). Since these states evolve unitarily their overlap at equal times is time independent, and is given by the usual expression: $\langle\alpha_2, t|\alpha_1, t\rangle = {}_0\langle\alpha_2|\alpha_1\rangle_0 = \mathrm{e}^{-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \bar\alpha_2\alpha_1}$.

4.4. $\mathfrak{su}(1, 1)$ *symmetry and coherent states*

As a consequence of the boson commutation relations for $d = 1$, the operators

$$K_0 = \tfrac{1}{4}(a^\dagger a + aa^\dagger), \qquad K_+ = \tfrac{1}{2}a^{\dagger 2}, \qquad K_- = \tfrac{1}{2}a^2 \tag{50}$$

satisfy the commutation relations of $\mathfrak{su}(1, 1)$:

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0, \tag{51}$$

and also form a set of dynamical invariants. Irreducible unitary representations of $\mathfrak{su}(1,1)$ are infinite dimensional and for the case at hand belong to the positive discrete series as discussed for example by Biedenharn and Louck [82] (p 276), also Perelomov [21] (p 70), and Kastrup [97]. States in the representation space are labeled by the eigenvalues of the commuting generators K_0, C where the Casimir invariant is

$$C = K_0^2 - \tfrac{1}{2}(K_-K_+ + K_+K_-) = K_0(K_0 - 1) - K_+K_-.$$

In an orthonormal basis $\{|\kappa, n\rangle\}$ where $n = 0, 1, 2, \dots$ we have the matrix elements (see for example Barut and Girardello [98]):

$$\begin{aligned} C|\kappa, n\rangle &= \kappa(\kappa - 1)|\kappa, n\rangle \\ K_0|\kappa, n\rangle &= (\kappa + n)|\kappa, n\rangle \\ K_+|\kappa, n\rangle &= \sqrt{(n+1)(n+2\kappa)}|\kappa, n+1\rangle \\ K_-|\kappa, n\rangle &= \sqrt{n(n+2\kappa-1)}|\kappa, n-1\rangle, \end{aligned} \tag{52}$$

where the Bargmann index κ is any positive number, for unitary representations of the covering group of SU(1, 1). For $d = 1$ the operators (50) correspond to the representations $\kappa = \frac{1}{4}$ (even states) or $\kappa = \frac{3}{4}$ (odd states), as discussed in [21] section 5.2.

There are two well-known analytic representations of coherent states for $\mathfrak{su}(1,1)$, the first being Perelomov generalized coherent states [21] which are obtained using the displacement operator formalism, the second being Barut–Girardello coherent states [98] which are eigenstates of the operator K_-. Since the states $|\alpha, t\rangle$ satisfy (46) we have $K_-|\alpha, t\rangle = \frac{1}{2}\alpha^2|\alpha, t\rangle$ and hence the time-dependent coherent states (41) are also Barut–Girardello coherent states. We construct radial $\mathfrak{su}(1,1)$ time-dependent coherent states in dimension d which satisfy the TDSE.

The creation and annihilation operators (45) extend to vector operators $\boldsymbol{a}, \boldsymbol{a}^\dagger$ in dimension d, from which $\mathfrak{su}(1,1)$ generators $K_\pm, K_0$ may be defined as the sum of d one-dimensional generators. One can define radial creation and annihilation operators by following the formalism developed in [99, 100], used to describe fractional dimensional quantum mechanics, in which the radial momentum and position operators P, Q, act on radial functions that depend on the angular momentum ℓ, and satisfy the R-deformed commutation relations as described in [99]. We proceed more directly, however, by defining the operator

$$K_- = \frac{\mathrm{e}^{2\mathrm{i}\omega_0\tau}}{4\hbar}\left[-\frac{\rho^2}{m\omega_0}\boldsymbol{p}^2 + \left(\mathrm{i} + \frac{\rho\dot\rho}{\omega_0}\right)(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}) + m\left(\frac{\omega_0}{\rho^2} - \frac{\dot\rho^2}{\omega_0} - \frac{2\mathrm{i}\dot\rho}{\rho}\right)\boldsymbol{q}^2\right], \tag{53}$$

together with $K_+ = K_-^\dagger$, $K_0 = I/(2\hbar\omega_0)$, where the invariant I is defined in (36). Then the operators $\{K_\pm, K_0\}$ are, firstly, dynamical invariants for the harmonic oscillator Hamiltonian (23) and, secondly, satisfy the commutation relations (51) of $\mathfrak{su}(1,1)$, as follows from their construction as quadratic combinations of the vector operators $\boldsymbol{a}, \boldsymbol{a}^\dagger$.

The operators $\{K_0, K_\pm\}$ act on the orthonormal basis vectors $\langle \boldsymbol{x}, t|\kappa, n\rangle$, which are a product of the radial functions $\psi_{n,\ell}(r,t) = \langle r, t|\kappa, n\rangle$ shown in (38) and time-independent angular functions. By comparing the eigenvalues of $I = 2\hbar\omega_0 K_0$ in (37) with those of K_0 we find that the $\mathfrak{su}(1,1)$ representation label κ is given by

$$\kappa = \tfrac{1}{4}(2\ell + d). \tag{54}$$

Hence $K_\pm, K_0$ act on the solutions $\psi_{n,\ell}(r,t)$ of the radial TDSE with matrix elements as shown in (52), either as raising or lowering operators (in the case of $K_\pm$), or diagonally in the case of K_0.

We construct firstly Barut–Girardello [98] coherent states $|z, \kappa\rangle_{\mathrm{BG}}$ which are the analogue of harmonic oscillator coherent states, namely eigenstates of the lowering operator K_-, that is, $K_-|z,\kappa\rangle_{\mathrm{BG}} = z|z,\kappa\rangle_{\mathrm{BG}}$ for any complex z. These states may be expressed as a linear superposition of the orthonormal states $|\kappa, n\rangle$, explicitly:

$$|z,\kappa\rangle_{\mathrm{BG}} = \frac{z^{\kappa-\frac{1}{2}}}{\sqrt{I_{2\kappa-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n+2\kappa)}}|\kappa, n\rangle, \tag{55}$$

where $I_{2\kappa-1}$ denotes a modified Bessel function of the first kind. Various properties of Barut–Girardello states including their overlap and completeness relations have been discussed in [101, 102] (and other references cited therein).

We evaluate the time-dependent states $\psi_{z,\ell}(r,t) = \langle r,t|z,\kappa\rangle_{\mathrm{BG}}$ by replacing the eigenstates $|\kappa,n\rangle$ by the normalized wavefunctions $\psi_{n,\ell}(r,t)$ given in (38), with κ given by (54). The sum over n can be performed with the help of the generating function for generalized Laguerre polynomials ([83], p 784), to obtain

$$\begin{aligned}\psi_{z,\ell}(r,t) &= \rho^{-1} r^{1-\frac{d}{2}}\,\mathrm{e}^{-\mathrm{i}\omega_0\tau}\exp(-z\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})\exp\left(\frac{\mathrm{i}m\dot\rho r^2}{2\hbar\rho}\right)\exp\left(-\frac{m\omega_0 r^2}{2\hbar\rho^2}\right)\\ &\quad\times I_{2\kappa-1}\left(\frac{2r\,\mathrm{e}^{-\mathrm{i}\omega_0\tau}}{\rho}\sqrt{\frac{zm\omega_0}{\hbar}}\right)\sqrt{\frac{m\omega_0}{\hbar I_{2\kappa-1}(2|z|)}}.\end{aligned} \tag{56}$$

As is evident from (55), these time-dependent Barut–Girardello coherent states are also solutions of the radial TDSE. They have parity $(-1)^\ell$ and for $d=1$, with $z=\frac{1}{2}\alpha^2$, reduce for $\ell=0$ to the even part of expression (48), and to the odd part for $\ell=1$, and they evolve unitarily from the initial wavefunction $\psi^0_{z,\ell}$ at time $t=t_0$ according to $\psi_{z,\ell}(r,t) = U(t,t_0)\psi_{z,\ell}(r,t_0)$. Radial Barut–Girardello coherent states have been investigated in [103] but not with a time evolution determined by the TDSE.

Perelomov generalized coherent states $|\zeta,\kappa\rangle_{\mathrm{P}}$ (described by Perelomov [21], chapter 5, also [101]) are obtained by allowing the unitary operator $\mathrm{e}^{\xi K_+ - \bar\xi K_-}$ to act on the vacuum, where ξ is a complex number. By expanding this operator one obtains ([21], equation (5.2.11)):

$$|\zeta,\kappa\rangle_{\mathrm{P}} = \mathrm{e}^{\xi K_+ - \bar\xi K_-}|\kappa,0\rangle = (1-|\zeta|^2)^\kappa \sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(n+2\kappa)}{n!\Gamma(2\kappa)}}\zeta^n|\kappa,n\rangle \tag{57}$$

where $\zeta = \frac{\xi}{|\xi|}\tanh|\xi|$ satisfies $|\zeta|<1$. The representation

$$|\zeta,\kappa\rangle_{\mathrm{P}} = (1-|\zeta|^2)^\kappa\,\mathrm{e}^{\zeta K_+}|\kappa,0\rangle \tag{58}$$

shows that these states can be identified as SU(1, 1) vector coherent states extensively used to construct representations of compact groups [104, 105] with extensions to quantum groups [106].

We explicitly evaluate the radial Perelomov coherent states $\psi_{\zeta,\ell}(r,t) = \langle r,t|\zeta,\kappa\rangle_{\mathrm{P}}$ in the coordinate representation by substituting the value κ shown in (54) and by replacing the eigenstates $|\kappa,n\rangle$ by the normalized wavefunctions $\psi_{n,\ell}(r,t)$ given in (38). The sum over n in (57) can again be performed explicitly with the help of the generating function for generalized Laguerre polynomials (see [83], p 784), to obtain

$$\begin{aligned}\psi_{\zeta,\ell}(r,t) &= \frac{r^\ell(1-|\zeta|^2)^\kappa}{\sqrt{\Gamma(2\kappa)}}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega_0\tau}}{1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau}}\right)^{2\kappa}\left(\frac{m\omega_0}{\hbar\rho^2}\right)^\kappa\exp\left(\frac{\mathrm{i}m\dot\rho r^2}{2\hbar\rho}\right)\\ &\quad\times\exp\left[-\frac{m\omega_0 r^2(1-\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})}{2\hbar\rho^2(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})}\right].\end{aligned} \tag{59}$$

These states satisfy the radial TDSE for the harmonic oscillator Hamiltonian (23) in d dimensions, and are normalized to unity as functions of r for any ζ in the unit disk $|\zeta|<1$ in the complex ζ-plane, with respect to the inner product (39). They have parity $(-1)^\ell$.

In contrast to the Barut–Girardello states (56), the functional form of the Perelomov states (59) does not change as d varies. Radial coherent states of the Perelomov type have been investigated in [107] but only with a time evolution corresponding to the time-independent harmonic oscillator.

The Perelomov states (59) evolve unitarily and so it follows, for example, that these states resolve the identity operator for $\kappa > \frac{1}{2}$, by integration over the unit disk in the complex ζ-plane [21, 101, 102]. Also, as a consequence, the overlap of two Perelomov coherent states at equal times is time independent and takes the known form:

$$ {}_{\rm P}\langle\zeta_1,\kappa|\zeta_2,\kappa\rangle_{\rm P} = \frac{(1-|\zeta_1|^2)^\kappa(1-|\zeta_2|^2)^\kappa}{(1-\overline{\zeta_1}\zeta_2)^{2\kappa}}, \tag{60} $$

as may be verified directly from expression (59).

The states (59) are eigenfunctions of two independent dynamical invariants, each parametrized by ζ. Let

$$ K_\zeta^0 = \frac{1}{1-|\zeta|^2}[(1+|\zeta|^2)K_0 - \zeta K_+ - \overline{\zeta}K_-], \tag{61} $$

where $K_\pm$, K_0 are defined by (53), then K_ζ^0 is a Hermitean dynamical invariant for the Hamiltonian (23) and the coherent states $\psi_{\zeta,\ell}$, comprising the radial functions (59) multiplied by the appropriate time-independent angular components, are eigenfunctions of K_ζ^0 (as discussed in [21], section 5.2.1 and [102], sections 7.2, 7.3), as may be verified directly:

$$ K_\zeta^0\psi_{\zeta,\ell} = \kappa\psi_{\zeta,\ell}, \tag{62} $$

where κ is defined in (54). Define also the non-Hermitean dynamical invariant

$$ K_\zeta^- = \frac{1}{1-|\zeta|^2}(-2\zeta K_0 + \zeta^2K_+ + K_-), \tag{63} $$

then the coherent state $\psi_{\zeta,\ell}$ lies in the kernel of K_ζ^-, i.e. $K_\zeta^-\psi_{\zeta,\ell} = 0$, which follows [21, 102] as a consequence of the $\mathfrak{su}(1,1)$ algebra satisfied by the invariants $K_\pm$, K_0. Define also

$$ K_\zeta^+ = \frac{1}{1-|\zeta|^2}(-2\overline{\zeta}K_0 + K_+ + \overline{\zeta}^2K_-) \tag{64} $$

then the operators $\{K_\zeta^0, K_\zeta^\pm\}$ generate $\mathfrak{su}(1,1)$, as follows from the fact that these operators are related by a linear SU(1, 1) transformation to the generators $\{K_0, K_\pm\}$.

Hence the state (59) for fixed ℓ,ζ may be regarded as the state of lowest weight in an $\mathfrak{su}(1,1)$ representation with the Bargmann index $\kappa = \frac{1}{4}(2\ell+d)$, with matrix elements as given in (52). Denote $|\kappa,0\rangle_\zeta = |\zeta,\kappa\rangle_{\rm P}$ then the normalized states

$$ |\kappa,n\rangle_\zeta = \left[\frac{\Gamma(2\kappa)}{n!\Gamma(n+2\kappa)}\right]^{1/2}\left(K_\zeta^+\right)^n|\kappa,0\rangle_\zeta \tag{65} $$

are eigenfunctions of K_ζ^0 with eigenvalues $\kappa+n$, and also satisfy the radial TDSE. We can compute these solutions, which may be termed excited coherent states, explicitly by observing that the quadratic invariant $K_\zeta^0[\rho]$, which is defined by (61) and may be expressed as a functional of ρ, is related to the usual quadratic invariant $I[\rho]$ as given in (36) according to $2\hbar\omega_0K_\zeta^0[\rho] = I[\rho_\zeta]$, where ρ_ζ is given in terms of ρ by the ζ-transformation (28) which leaves the Ermakov equation (25) invariant. Hence, the eigenfunctions of $K_\zeta^0[\rho]$ may be determined by replacing $\rho\to\rho_\zeta$ in the explicit functions $\psi_{n,\ell}(r,t)$ given in (38). Furthermore, these eigenfunctions also satisfy the radial TDSE, since ρ_ζ satisfies the Ermakov equation and, up to a constant phase, are identical to the wavefunctions $\psi_{n,\zeta,\ell}(r,t) = \langle r,t|\kappa,n\rangle_\zeta$ defined by (65).

By means of (30) we therefore derive the following explicit solutions of the radial TDSE:

$$\psi_{n,\zeta,\ell}(r,t) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+2\kappa)}} \left(\frac{m\omega_0}{\hbar\rho_\zeta^2}\right)^{\kappa} \exp\left(\frac{\mathrm{i}m\dot{\rho}_\zeta r^2}{2\hbar\rho_\zeta}\right) \exp\left(-\frac{m\omega_0 r^2}{2\hbar\rho_\zeta^2}\right)$$
$$\times r^\ell L_n^{(2\kappa-1)}\left(\frac{m\omega_0 r^2}{\hbar\rho_\zeta^2}\right) \left[\frac{(1+\bar{\zeta}\,\mathrm{e}^{2\mathrm{i}\omega_0\tau})\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})}{1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau}}\right]^{n+\kappa}, \tag{66}$$

where in summary we have n, ℓ are non-negative integers, ζ is any complex number with $|\zeta| < 1, 4\kappa = 2\ell + d$, L denotes generalized Laguerre polynomials, ρ_ζ is defined in terms of ρ by (28) where ρ is any solution of the Ermakov equation (25), τ is given by (21) and ω_0 is an arbitrary positive constant which can be set to unity. For $\zeta = 0$ these states reduce to the wavefunctions (38), and for $n = 0$ reduce to the Perelomov coherent states (59). The states (66) are normalized to unity with respect to the inner product (39), evaluated at equal times for fixed parameters n, ℓ, ζ, and are orthogonal for different values of n for fixed ℓ, ζ.

We see therefore that time-dependent Perelomov generalized coherent states, also known as vector coherent states, are related to the time-dependent wavefunctions (38) by the ζ-transformation (28), which is implemented by means of a unitary transformation on the wavefunction as expressed in the representations (57) or (58). These expressions also show how the general solution (66) of the TDSE can be represented as a function of invariant operators acting on the ground (vacuum) state.

For $d = 1$, with $\ell = 0, \kappa = \frac{1}{4}$, the ground state $\langle x, t|0\rangle_\zeta$ is given by the coherent state (59) and the excited states $\psi_{n,\zeta}(x,t) = \langle x, t|n\rangle_\zeta$ have the following explicit form:

$$\psi_{n,\zeta}(x,t) = \left[\frac{m\omega_0(1-|\zeta|^2)\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau}}{2^{2n}n!^2\pi\hbar\rho^2(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})^2}\right]^{\frac{1}{4}} \left[\frac{(1+\bar{\zeta}\,\mathrm{e}^{2\mathrm{i}\omega_0\tau})\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau}}{(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})}\right]^{\frac{n}{2}} \exp\left(\frac{\mathrm{i}m\dot{\rho}x^2}{2\hbar\rho}\right)$$
$$\times \exp\left[-\frac{m\omega_0 x^2(1-\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})}{2\hbar\rho^2(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})}\right] H_n\left(\frac{x}{\rho}\sqrt{\frac{m\omega_0(1-|\zeta|^2)}{\hbar(1+\zeta\,\mathrm{e}^{-2\mathrm{i}\omega_0\tau})(1+\bar{\zeta}\,\mathrm{e}^{2\mathrm{i}\omega_0\tau})}}\right). \tag{67}$$

As before, n is a non-negative integer, ζ is any complex number with $|\zeta| < 1$, H denotes Hermite polynomials, τ is given by (21) and ω_0 is an arbitrary positive constant which can be set to unity. These states satisfy the TDSE and for $\zeta = 0$ reduce to the solutions (41).

The ζ-dependence of the states (67) can be analyzed by defining the boson pair $\left(a_\zeta, a_\zeta^\dagger\right)$ obtained by replacing $\rho \to \rho_\zeta$ in the definition (45), where ρ_ζ is given by (28), i.e. we define $a_\zeta = a[\rho_\zeta], a_\zeta^\dagger = a^\dagger[\rho_\zeta]$. This boson pair is related linearly to $(a, a^\dagger)$ by the SU(1, 1) transformation g_ζ defined in (32), as follows from (34), hence

$$a_\zeta = \frac{\mathrm{e}^{-\mathrm{i}\delta}}{\sqrt{1-|\zeta|^2}}(a - \zeta a^\dagger), \qquad a_\zeta^\dagger = \frac{\mathrm{e}^{\mathrm{i}\delta}}{\sqrt{1-|\zeta|^2}}(a^\dagger - \bar{\zeta}a), \tag{68}$$

where the phase factor $\mathrm{e}^{\mathrm{i}\delta}$ is defined by (33). The operators

$$K_\zeta^0 = \tfrac{1}{4}\left(a_\zeta^\dagger a_\zeta + a_\zeta a_\zeta^\dagger\right), \qquad K_\zeta^+ = \tfrac{1}{2}\left(a_\zeta^\dagger\right)^2, \qquad K_\zeta^- = \tfrac{1}{2}a_\zeta^2$$

agree with those defined in (61), (63) and (64), up to a constant phase, where $K_0, K_\pm$ are given by (50). The ζ-ground state $|0\rangle_\zeta$ is given in the coordinate representation by (59) for $d = 1, \ell = 0, \kappa = \frac{1}{4}$. By means of (68) we find, by solving $a_\zeta|0\rangle_\zeta = 0$, that

$$|0\rangle_\zeta = (1-|\zeta|^2)^{\frac{1}{4}} \exp\left[\frac{\zeta(a^\dagger)^2}{2}\right]|0\rangle, \tag{69}$$

in accordance with (58). The excited states are given by

$$|n\rangle_\zeta = \frac{(a_\zeta^\dagger)^n}{\sqrt{n!}}|0\rangle_\zeta = \frac{e^{in\delta}(1-|\zeta|^2)^{\frac{1}{4}}}{\sqrt{n!}}\left(\frac{\bar{\zeta}}{2}\right)^{\frac{n}{2}} H_n\left(a^\dagger\sqrt{\frac{1-|\zeta|^2}{2\bar{\zeta}}}\right)\exp\left[\frac{\zeta(a^\dagger)^2}{2}\right]|0\rangle, \tag{70}$$

where H is a Hermite polynomial, as may be proved by induction on n using $a_\zeta^\dagger|n\rangle_\zeta = \sqrt{n+1}|n+1\rangle_\zeta$. The overlap ${}_{\zeta_1}\langle m|n\rangle_{\zeta_2}$ is generally nonzero for arbitrary ζ_1, ζ_2.

While the ζ-transformation $\rho \to \rho_\zeta$ is of particular interest because it relates Perelomov coherent states and their excitations to the eigenfunctions of the Lewis–Riesenfeld invariant I, it can also be applied to other solutions of the TDSE, such as the Barut–Girardello coherent states (56) to obtain general solutions $\psi_{z,\ell,\zeta}(r,t)$. Similarly the coherent states (48) can be generalized to states $\psi_{\alpha,\zeta}(x,t)$, and corresponding properties follow by replacing $\rho \to \rho_\zeta$. In general the ζ-transformation is performed on wavefunctions by an explicit unitary transformation.

4.5. Algebra of invariants for the harmonic oscillator

The set of dynamical invariants for a given Hamiltonian forms an algebra, i.e. for any two invariants I_1, I_2 the product I_1I_2 and the linear combination $c_1I_1 + c_2I_2$ for any constants c_1, c_2 are also invariants. We investigate explicitly here the algebra of the linear and quadratic invariants for the harmonic oscillator, which are functionals of ρ, and for fixed ρ satisfy the inhomogeneous $\mathfrak{su}(1,1)$ algebra. For general ρ satisfying the Ermakov equation these invariants depend on two parameters, essentially the initial values $\rho(t_0), \dot{\rho}(t_0)$, but which for convenience we take to be the complex parameter ζ discussed in the context of the ζ-transformation (28). We explicitly express the sum and product of two invariants with parameters ζ_1, ζ_2 as a third invariant with a parameter ζ_3, i.e. we relate the sum and product of $I[\rho_1], I[\rho_2]$ to $I[\rho]$ for some ρ.

Two invariants $I[\rho_1]$ and $I[\rho_2]$ are related by a unitary transformation as may be determined from expression (14) for $I[\rho]$. We have $I = T[\rho]H^0T^\dagger[\rho]$ where $T[\rho] = T_{\mathrm{q}}[\rho]T_{\mathrm{dil}}[\rho]$ and $T_{\mathrm{dil}}, T_{\mathrm{q}}$ are given by (11) and (13) with $\alpha = m\dot{\rho}/\rho$, hence

$$I[\rho_2] = T[\rho_2]T^\dagger[\rho_1]I[\rho_1]T[\rho_1]T^\dagger[\rho_2].$$

Wavefunctions ψ_1 which are solutions of the TDSE and eigenfunctions of $I[\rho_1]$ are related to the wavefunctions ψ_2 of $I[\rho_2]$ by a corresponding unitary transformation: $\psi_2 = T[\rho_2]T^\dagger[\rho_1]\psi_1$. Since ρ_1, ρ_2 are related by the ζ-transformation we may explicitly evaluate the unitary operator $T_\zeta = T[\rho_2]T^\dagger[\rho_1]$ as a function of ζ.

Consider now the Hermitean quadratic invariant $I[\rho]$ defined by (36), then for any solutions ρ_1, ρ_2 of the Ermakov equation (25) there exists a constant c such that

$$cI[\rho] = c_1I[\rho_1] + c_2I[\rho_2],$$

where ρ, defined in terms of ρ_1, ρ_2 by the addition formula $c\rho^2 = c_1\rho_1^2 + c_2\rho_2^2$, also satisfies the Ermakov equation (25). We find

$$c^2 = (c_1 - c_2)^2 + \frac{4c_1c_2}{1-|\zeta|^2}$$

where ζ is the parameter such that ρ_2 is the ζ-transform of ρ_1. The commutator of two Hermitean quadratic invariants $I[\rho_1]$ and $I[\rho_2]$ is also a quadratic invariant, specifically $[I[\rho_1], I[\rho_2]] = 2i\hbar c' I[\rho]$ where ρ, defined by $c'\rho^2 = \rho_1^2\rho_2\dot{\rho}_2 - \rho_1\dot{\rho}_1\rho_2^2$, satisfies the Ermakov equation (25) except with ω_0^2 replaced by $-\omega_0^2$. The constant is given by $c' = 2\omega_0|\zeta|/(1-|\zeta|^2)$.

Commutators amongst the non-Hermitean quadratic invariants $K_\pm[\rho_1]$ and $K_\pm[\rho_2]$ defined in (53), as well as $K_0[\rho_1], K_0[\rho_2]$, are also quadratic in momentum and position and are conveniently investigated via the boson operators $a^\dagger[\rho]$, $a[\rho]$ to which they are related for $d = 1$ by (50). As before, there exists ζ such that ρ_2 is the ζ-transform of ρ_1, then from (68) we find

$$[a[\rho_1], a[\rho_2]] = -\frac{\zeta\, \mathrm{e}^{-\mathrm{i}\delta}}{\sqrt{1-|\zeta|^2}}\mathbb{I}, \tag{71}$$

and also

$$[a[\rho_1], a^\dagger[\rho_2]] = \frac{\mathrm{e}^{\mathrm{i}\delta}}{\sqrt{1-|\zeta|^2}}\mathbb{I}, \tag{72}$$

where $\mathrm{e}^{\mathrm{i}\delta}$ is given by (33). Linear combinations of $a, a^\dagger$ for different functions ρ_1, ρ_2 are also linear in p, q and hence can be expressed as a linear combination of $a[\rho]$ and $a^\dagger[\rho]$ for any given ρ. For example, there exist constants z_1, z_2 such that for any ρ_1, ρ_2, ρ and constants c_1, c_2 we have $c_1 a[\rho_1] + c_2 a[\rho_2] = z_1 a[\rho] + z_2 a^\dagger[\rho]$, where z_1, z_2 can be determined using (71) and (72). Similarly, the product of any two creation or annihilation operators $a, a^\dagger$ for different functions ρ_1, ρ_2 is an invariant which is quadratic in the operators p, q, and so can be expressed in terms of the invariants $K_\pm[\rho]$, $K_0[\rho]$ for any ρ satisfying the Ermakov equation. By means of (68) we find

$$a[\rho_1]a[\rho_2] = \frac{2\,\mathrm{e}^{-\mathrm{i}(\delta_1+\delta_2)}}{\sqrt{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}}\left[\frac{1}{4}(\zeta_1-\zeta_2)\mathbb{I} + K_- + \zeta_1\zeta_2 K_+ - 2(\zeta_1+\zeta_2)K_0\right], \tag{73}$$

where ρ_1, ρ_2 are each related to ρ by the ζ-transformation (28) with corresponding parameters ζ_1, ζ_2, and phases δ_1, δ_2 given by (33). Similar relations hold for the product of any pair of creation and annihilation operators for different functions ρ, from which one can establish commutators amongst any pair of $K_\pm, K_0$ for different ρ, and so obtain formulae which are valid for any d.

4.6. Linear Hermitean invariant, and plane wave and δ-function solutions

The invariants $K_\pm, K_0$ in (53) are quadratic in both momentum and position operators, whereas the boson invariants $a, a^\dagger$ defined in (45) for $d = 1$ are linear in p, q, in particular the Hermitean invariant $\mathrm{i}(a^\dagger - a)$ is linear in both p, q, and may be used to find plane wave solutions of the TDSE. The operator $\mathrm{i}(a_0^\dagger - a_0) = \mathrm{i}U^\dagger(a^\dagger - a)U$ is proportional to the momentum operator p, the eigenfunctions of which are the non-normalizable plane waves $\phi_k^0(x) = \mathrm{e}^{\mathrm{i}kx}$, with real eigenvalues $\lambda = \hbar k$, which evolve according to $\psi_k = U\phi_k^0$. A general solution ψ can therefore be constructed by expanding a general initial wavefunction ψ^0 as a Fourier series, and ψ then appears as an integral transform of the solutions ψ_k. The construction of solutions by means of integral transforms has been investigated in [78], and as a Fourier transform by Hartley and Ray [79] but ignoring singularities which, as we discuss below, signify the appearance of distributional (i.e., δ-function) solutions.

The time evolution operator is given by (40) where for ρ we choose the initial conditions $\rho(t_0) = 1, \dot{\rho}(t_0) = 0$, in which case $T(t_0) = \mathbb{I}$. We explicitly evaluate $\psi_k = T_{\mathrm{q}}\, T_{\mathrm{dil}} \exp(-\mathrm{i}\tau H^0/\hbar)\phi_k^0$ by first computing the factor $\exp(-\mathrm{i}\tau H^0/\hbar)\phi_k^0$ using a BCH (Baker–Campbell–Haussdorf) formula, see for example the discussion in [108–110]. We have in general

$$\mathrm{e}^{\mathrm{i}a^2 J_+ + \mathrm{i}b^2 J_-} = \mathrm{e}^{\alpha J_0}\, \mathrm{e}^{\mathrm{i}\beta J_+}\, \mathrm{e}^{\mathrm{i}\gamma J_-}, \tag{74}$$

where $\{J_\pm, J_0\}$ generates the Lie algebra $\mathfrak{sl}_2$, i.e. $[J_-, J_+] = 2J_0$ and $[J_0, J_\pm] = \pm J_\pm$, and the parameters α, β, γ are given in terms of a, b by

$$\alpha = -\log\cos^2 ab, \qquad \beta = \frac{a}{2b}\sin 2ab, \qquad \gamma = \frac{b}{a}\tan ab. \tag{75}$$

If we identify the $\mathfrak{sl}_2$ generators by

$$J_+ = -\frac{q^2}{2\hbar}, \qquad J_- = -\frac{p^2}{2\hbar}, \qquad J_0 = \frac{\mathrm{i}}{4\hbar}(pq + qp), \tag{76}$$

then we find

$$\exp\left(-\frac{\mathrm{i}\tau}{\hbar}H^0\right) = \exp\left[-\frac{\mathrm{i}\log\cos^2\omega_0\tau}{4\hbar}(pq+qp)\right]\exp\left(-\frac{\mathrm{i}m\omega_0\sin 2\omega_0\tau}{4\hbar}q^2\right)$$
$$\times\exp\left(-\frac{\mathrm{i}\tan\omega_0\tau}{2m\omega_0\hbar}p^2\right), \tag{77}$$

and so the solutions $\psi_k(x,t) = U(t,t_0)\phi_k^0(x)$ of the TDSE, corresponding to the initial wavefunctions $\mathrm{e}^{\mathrm{i}kx}$, are given by

$$\psi_k(x,t) = (\rho\cos\omega_0\tau)^{-\frac{1}{2}}\exp\left(\frac{\mathrm{i}m\dot{\rho}x^2}{2\hbar\rho}\right)\exp\left(-\frac{\mathrm{i}m\omega_0x^2\tan\omega_0\tau}{2\hbar\rho^2}\right)$$
$$\times\exp\left(-\frac{\mathrm{i}\hbar k^2\tan\omega_0\tau}{2m\omega_0}\right)\exp\left(\frac{\mathrm{i}kx}{\rho\cos\omega_0\tau}\right), \tag{78}$$

for any real k.

Let $\psi^0(x)$ be any initial wavefunction with a Fourier transform $\widehat{\psi}_0$:

$$\psi^0(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{\psi}_0(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{79}$$

then the unique solution of the TDSE satisfying the initial condition $\psi(x,t_0) = \psi^0(x)$ is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{\psi}_0(k)\psi_k(x,t)\,\mathrm{d}k, \tag{80}$$

where $\psi_k(x,t)$ is given by (78). If ψ^0 is normalizable then by Parseval's theorem $\widehat{\psi}_0(k)$ is also normalizable, as is the solution $\psi_k(x,t)$ at time t. For example, if $\psi^0(x)$ is the Gaussian wavefunction corresponding to the ground state $n = 0$ of the time-dependent solution (41) at the initial time $t = t_0$, namely

$$\psi^0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{m\omega_0x^2}{2\hbar}\right),$$

then the Fourier transform is

$$\widehat{\psi}_0(k) = \left(\frac{\hbar}{\pi m\omega_0}\right)^{\frac{1}{4}}\exp\left(-\frac{k^2\hbar}{2m\omega_0}\right), \tag{81}$$

and the solution at time t, evaluated by computing the integral (80), is given by (41) for $n = 0$.

The plane wave solutions (78) may also be used to construct a distributional solution, i.e. the solution $\delta(x,t)$ of the TDSE for which the initial function is the Dirac δ-function, $\delta(x,t_0) = \delta(x)$. Such a solution provides another means of expressing the general solution of the TDSE as an integral transform, according to

$$\psi(x,t) = \int_{-\infty}^{\infty}\delta(x-y,t)\psi^0(y)\,\mathrm{d}y \tag{82}$$

where ψ^0 is any initial wavefunction. From the representation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$$

we determine, by choosing $\widehat{\psi}_0(k) = 1/\sqrt{2\pi}$ in (80), the following distributional solution of the TDSE:

$$\delta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_k(x,t)\,\mathrm{d}k \tag{83}$$

$$= \mathrm{e}^{-\mathrm{i}\pi/4}\left(\frac{m\omega_0}{2\pi\hbar\rho\sin\omega_0\tau}\right)^{\frac{1}{2}}\exp\left(\frac{\mathrm{i}m\dot\rho x^2}{2\hbar\rho}\right)\exp\left(\frac{\mathrm{i}m\omega_0x^2}{2\hbar\rho^2\tan\omega_0\tau}\right), \tag{84}$$

where we have substituted for $\psi_k(x,t)$ from (78) and used the Fresnel integral

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}ky}\,\mathrm{e}^{-\mathrm{i}ak^2}\,\mathrm{d}k = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{\pi}{a}}\exp\left(\frac{\mathrm{i}y^2}{4a}\right), \qquad a>0. \tag{85}$$

The fact that the solution (84) reduces to $\delta(x)$ at $t=t_0$ can be verified by first defining the following complex function, an approximate δ-function, for any $\varepsilon>0$:

$$\delta^\varepsilon(x) = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{4\pi\varepsilon}}\exp\left(\frac{\mathrm{i}x^2}{4\varepsilon}\right), \tag{86}$$

then from (85) we have $\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{e}^{-\mathrm{i}\varepsilon k^2}\,\mathrm{d}k = 2\pi\delta^\varepsilon(x)$ and also, from the complex conjugate of (85), $\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\delta^\varepsilon(x)\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\varepsilon k^2}$. In particular, at $k=0$ we have $\int_{-\infty}^{\infty}\delta^\varepsilon(x)\,\mathrm{d}x = 1$ for every $\varepsilon>0$. More generally, for any function f with a Fourier transform $\widehat{f}$ we have formally:

$$\int_{-\infty}^{\infty}f(x)\delta^\varepsilon(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\delta^\varepsilon(x)\,\mathrm{d}x\right)\mathrm{d}k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)\,\mathrm{e}^{-\mathrm{i}\varepsilon k^2}\mathrm{d}k.$$

Hence, as $\varepsilon\to0$ we have $\int_{-\infty}^{\infty}f(x)\delta^\varepsilon(x)\,\mathrm{d}x \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(k)\,\mathrm{d}k = f(0)$ and so in the sense of distributions we have $\delta^\varepsilon(x)\to\delta(x)$ as $\varepsilon\to0$ (see for example chapter 12 of [80]). We find from the solution (84) that $\delta(x,t)/\delta^\varepsilon(x)\to1$ as $t\to t_0$ where $\varepsilon=(t-t_0)\hbar/(2m)$, which demonstrates that indeed the solution (84) satisfies $\delta(x,t_0)=\delta(x)$.

Generally, the solutions (78) and (84) exist as functions only within restricted time intervals due to the appearance of zeros in the denominator. The factors $f=\rho\cos\omega_0\tau$ or $f=\rho\sin\omega_0\tau$ each satisfy the linear classical harmonic oscillator equation (26), as discussed in section 4.1, and if the given frequency $\omega(t)$ satisfies the condition (35) for example, then $\cos\omega_0\tau$ and $\sin\omega_0\tau$ each have an infinite number of zeros for $t>t_0$. The consequent singularities in the solutions (78) or (84) signify the appearance of δ-functions, as already demonstrated for (84) at $t=t_0$. In order to verify this more generally, consider (78) at the time $t=t_1$ for which $\omega_0\tau(t_1)=\pi/2$, i.e. at $t=t_1=\tau^{-1}(\pi/(2\omega_0))$. For small $t-t_1>0$ we have

$$\psi_k(x,t) = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{2\pi\hbar\rho(t_1)}{m\omega_0}}\exp\left(-\frac{\mathrm{i}m\dot\rho(t_1)x^2}{2\hbar\rho(t_1)}\right)\delta^\varepsilon\left(x-\frac{\hbar k\rho(t_1)}{m\omega_0}\right)+O(t-t_1)^{1/2},$$

where δ^ε is defined by (86), with $\varepsilon=(t-t_1)\hbar/(2m)$. Hence

$$\psi_k(x,t_1) = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{2\pi\hbar\rho(t_1)}{m\omega_0}}\exp\left(-\frac{\mathrm{i}m\dot\rho(t_1)x^2}{2\hbar\rho(t_1)}\right)\delta\left(x-\frac{\hbar k\rho(t_1)}{m\omega_0}\right),$$

and similarly at other zeros of $\cos\omega_0\tau$.

In d dimensions the general solution can be constructed by means of the d-dimensional Fourier transform, which for radial solutions reduces to a Hankel transform. We again re-order the operator $\exp(-\mathrm{i}\tau H^0/\hbar)$ by means of the BCH formula as shown in (77), and the eigenfunctions of $\boldsymbol{p}^2$ with eigenvalues $\hbar^2k^2$ are given by the product of time-independent angular components and the radial functions

$$\phi^0_{k,\ell}(r) = r^{1-\frac{d}{2}}J_{\ell-1+\frac{d}{2}}(kr), \tag{87}$$

where J denotes a Bessel function of the first kind. These functions are regular at the origin, with parity $(-1)^\ell$, and for $d = 1$ and $\ell = 0, 1$ reduce to the usual trigonometric functions. The solution of the radial TDSE is

$$\psi_{k,\ell}(r,t) = \frac{r^{1-\frac{d}{2}}}{\rho\cos\omega_0\tau} J_{\ell-1+\frac{d}{2}}\left(\frac{kr}{\rho\cos\omega_0\tau}\right)\exp\left(\frac{\mathrm{i}m\dot{\rho}r^2}{2\hbar\rho}\right) \times \exp\left(-\frac{\mathrm{i}m\omega_0 r^2\tan\omega_0\tau}{2\hbar\rho^2}\right)\exp\left(-\frac{\mathrm{i}\hbar k^2\tan\omega_0\tau}{2m\omega_0}\right), \tag{88}$$

and at the initial time reduces to the functions (87). The radial solution which evolves from an initial wavefunction $\psi_0(r) = \psi_\ell(r, t_0)$, which we assume carries definite angular momentum ℓ, is constructed by superposition:

$$\psi_\ell(r,t) = \int_0^\infty \widehat{\psi}_0(k)\sqrt{k}\psi_{k,\ell}(r,t)\,\mathrm{d}k, \tag{89}$$

where $\widehat{\psi}_0$ is related to the Hankel transform of $\psi_0(r)$. Specifically, the Hankel transform $\mathcal{H}_\nu[g](k)$ of a suitable function g is defined [112, 113] by

$$\mathcal{H}_\nu[g](k) = \int_0^\infty g(r)\sqrt{kr}J_\nu(kr)\,\mathrm{d}r,$$

and is a self-reciprocal integral transform. The function $\widehat{\psi}_0$ in (89) is given by

$$\widehat{\psi}_0(k) = \mathcal{H}_{\ell-1+\frac{d}{2}}\left[r^{\frac{1}{2}(d-1)}\psi_0(r)\right](k),$$

and so (89) is the unique solution of the radial TDSE satisfying the initial condition $\psi_\ell(r, t_0) = \psi_0(r)$.

5. Translations

Whereas in section 3 we considered time-dependent unitary transformations T which included dilatations, we now investigate the case where T comprises time-dependent translations in d dimensions. We choose H^0 as before in (10), with eigenfunctions $\phi^0_\lambda(\boldsymbol{x})$ and eigenvalues λ, and define

$$T = \mathrm{e}^{-\mathrm{i}m\gamma/\hbar}\exp\left(\frac{\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{q}}{\hbar}\right)\exp\left(-\frac{\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\hbar}\right), \tag{90}$$

where $\boldsymbol{\alpha}, \boldsymbol{\sigma}$ are any vector functions of time, and where $\gamma(t)$ is included for convenience. The quadratic invariant $I = TH^0T^\dagger$ is given by

$$I = \frac{1}{2m}(\boldsymbol{p}-\boldsymbol{\alpha})^2 + U(\boldsymbol{q}-\boldsymbol{\sigma}) \tag{91}$$

and we have $\mathrm{i}\hbar(\partial T/\partial t)T^\dagger = \dot{\boldsymbol{\sigma}}\cdot(\boldsymbol{p}-\boldsymbol{\alpha}) - \dot{\boldsymbol{\alpha}}\cdot\boldsymbol{q} + m\dot{\gamma}$. We choose $F(I,t) = I/\rho^2$ where ρ is strictly positive, then according to (5) we have the Hamiltonian

$$\mathcal{H} = \frac{1}{2m\rho^2}(\boldsymbol{p}-\boldsymbol{\alpha})^2 + \dot{\boldsymbol{\sigma}}\cdot(\boldsymbol{p}-\boldsymbol{\alpha}) - \dot{\boldsymbol{\alpha}}\cdot\boldsymbol{q} + \frac{1}{\rho^2}U(\boldsymbol{q}-\boldsymbol{\sigma}) + m\dot{\gamma},$$

in which the terms linear in $\boldsymbol{p}$ cancel provided $\boldsymbol{\alpha} = m\rho^2\dot{\boldsymbol{\sigma}}$. If we also choose $\rho = 1$ then we find $\mathcal{H}(t) = \boldsymbol{p}^2/2m + V(\boldsymbol{x}, t)$ with

$$V(\boldsymbol{x},t) = U(\boldsymbol{x}-\boldsymbol{\sigma}) - m\ddot{\boldsymbol{\sigma}}\cdot\boldsymbol{x} - \frac{m}{2}\dot{\boldsymbol{\sigma}}\cdot\dot{\boldsymbol{\sigma}} + m\dot{\gamma} \tag{92}$$

and an exact solution of the TDSE is

$$\psi_\lambda(\boldsymbol{x},t) = \mathrm{e}^{-\mathrm{i}\lambda(t-t_0)/\hbar}\,\mathrm{e}^{-\mathrm{i}\gamma/\hbar}\exp\left(\frac{\mathrm{i}m\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{x}}{\hbar}\right)\phi^0_\lambda(\boldsymbol{x}-\boldsymbol{\sigma}). \tag{93}$$

Various models with time-dependent terms may be investigated by means of time-dependent translations possibly combined with dilatations, for example the infinite square well for $d = 1$ with independently moving left and right walls has been considered in [45, 42, 46], also the harmonic oscillator with time-dependent linear terms, see for example [114, 21] (chapter 18).

The linear potential with a time-dependent coefficient has been extensively discussed, see for example Myers [115] (1960) and [116], also more recently [117–123]. In this case the time-dependent Hamiltonian (for $d = 1$) is

$$\mathcal{H} = \frac{p^2}{2m} + Z(t)x, \tag{94}$$

where Z is a given function of t, and solutions of the TDSE may be found by means of either translations or dilatations with the dilatation function ρ given by (19), as discussed in section 3.1. We demonstrate how, by suitable choice of H^0, we may find solutions in several forms, firstly plane wave solutions and hence the general solution expressed as an integral transform, using either the Fourier transform or the distributional solution. We also find solutions which are eigenfunctions of invariants which are quadratic in p and linear or quadratic in q, and also coherent state solutions including $\mathfrak{su}(1,1)$ coherent states.

We choose firstly $H^0 = p^2/(2m) + Z_0 x$ for constant Z_0, i.e. $U(x) = Z_0 x$ in (92) which together with $m\ddot{\sigma} = Z_0 - Z(t)$ and a suitable choice of γ gives $V(x,t) = Z(t)x$. The solutions (93) of the TDSE may be expressed in terms of Airy functions (as found in [117, 120, 124]) which appear as the eigenfunctions ϕ^0_λ of H^0, and the time evolution operator is given by (8), but can be re-arranged into the form (97) shown below.

We obtain a linear invariant by choosing $H^0 = p$, with T again given by (90) leading to $I = p - \alpha$ and then, by means of the formula (5) and the choice $F(I,t) = I^2/2m$, we obtain the Hamiltonian (94) provided $m\ddot{\sigma} = -Z(t)$ with $\alpha = m\dot{\sigma}$ and $2\dot{\gamma} = \dot{\sigma}^2$. The eigenfunctions of $H^0 = p$ are the plane waves $\phi^0_k(x) = \mathrm{e}^{\mathrm{i}kx}$ with eigenvalues $\lambda = \hbar k$ and so according to (6) the corresponding solutions ψ_k of the TDSE are given by

$$\psi_k(x,t) = \mathrm{e}^{-\mathrm{i}m\gamma/\hbar}\,\mathrm{e}^{\mathrm{i}m\dot{\sigma}x/\hbar}\,\mathrm{e}^{\mathrm{i}k(x-\sigma)}\exp\left[-\frac{\mathrm{i}\hbar k^2(t-t_0)}{2m}\right]. \tag{95}$$

We choose the initial values $\sigma(t_0) = 0 = \alpha(t_0) = \gamma(t_0)$ which implies $\dot{\sigma}(t_0) = 0$, specifically, given $Z(t)$ we define

$$\sigma(t) = \frac{1}{m}\int_{t_0}^{t}(s-t)Z(s)\,\mathrm{d}s, \tag{96}$$

with corresponding expressions for $\alpha(t)$ and $\gamma(t)$, then from (90) we have $T(t_0) = \mathbb{I}$ and the initial wavefunction is $\psi^0_k(x) = \psi_k(x,t_0) = \mathrm{e}^{\mathrm{i}kx}$. The general solution of the TDSE can be expressed as an integral transform by firstly writing ψ^0 as a Fourier transform, as shown in (79), and then the solution $\psi(x,t)$ at any later time is given by (80), where now $\psi_k(x,t)$ is given by (95). According to (8) the time evolution operator is given by

$$U(t,t_0) = \mathrm{e}^{-\mathrm{i}m\gamma/\hbar}\,\mathrm{e}^{\mathrm{i}m\dot{\sigma}q/\hbar}\,\mathrm{e}^{-\mathrm{i}\sigma p/\hbar}\exp\left[-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}\right], \tag{97}$$

which satisfies the TDSE with the initial condition $U(t_0,t_0) = \mathbb{I}$, and has a form which is well known [116, 124–126] for both time-independent and time-dependent cases.

Having determined the plane wave solutions (95) we may also compute the distributional solution $\delta(x,t)$ with the initial value $\delta(x)$, according to the formula (83). With the help of the integral (85) we find

$$\delta(x,t) = \mathrm{e}^{-\mathrm{i}m\gamma/\hbar}\,\mathrm{e}^{\mathrm{i}m\dot{\sigma}x/\hbar}\delta^{\varepsilon}(x-\sigma), \tag{98}$$

where δ^ε is defined in (86), with $\varepsilon = (t - t_0)\hbar/(2m)$. The function $\delta(x, t)$ satisfies the TDSE and at $t = t_0$ takes the value $\delta(x)$ as discussed following equation (85). The general solution of the TDSE, for any initial wavefunction $\psi^0(x)$, may now be represented as the integral transform (82).

Since $U(t, t_0)$ in (97) is an element of the inhomogeneous SU(1, 1) group, the invariant $I = U\mathcal{O}U^\dagger$, where $\mathcal{O}$ equals p, q or a quadratic combination of p, q, is likewise linear or quadratic in p, q, as is the case also for the harmonic oscillator. A second linear invariant, additional to $I = p - m\dot\sigma$, leads to coherent state solutions in which we choose the initial wavefunction to be the coherent state $\phi^0_\alpha(x)$ in (47), i.e. the eigenfunction of the boson annihilation operator a_0 shown in (44). In this case $H^0 = a_0$ and the corresponding invariant is the boson operator $a = Ua_0U^\dagger$ where U is given in (97). Explicitly

$$a = \sqrt{\frac{m\omega_0}{2\hbar}}\left[q - \sigma - \frac{t - t_0}{m}(p - m\dot\sigma)\right] + \frac{\mathrm{i}}{\sqrt{2m\omega_0\hbar}}(p - m\dot\sigma). \tag{99}$$

The time-dependent coherent state ψ_α which satisfies the TDSE is therefore given by $\psi_\alpha(x, t) = U(t, t_0)\phi^0_\alpha(x)$ and may be determined in several ways, for example directly by using (97), or by means of the formula (82) in which $\psi^0 = \phi^0_\alpha$ is the coherent state (47) and $\delta(x, t)$ is given by (98). The explicit solution, for any complex α, is

$$\psi_\alpha(x, t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac14} \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{-\alpha^2/2}\,\mathrm{e}^{-\mathrm{i}m\gamma/\hbar}\,\mathrm{e}^{\mathrm{i}m\dot\sigma x/\hbar}\,[1 + \mathrm{i}\omega_0(t - t_0)]^{-1/2}$$
$$\times \exp\left[\frac{\mathrm{i}\omega_0(t - t_0)\alpha^2}{1 + \mathrm{i}\omega_0(t - t_0)}\right]\exp\left[-\frac{m\omega_0(x - \sigma)^2}{2\hbar(1 + \mathrm{i}\omega_0(t - t_0))} + \frac{\alpha(x - \sigma)}{1 + \mathrm{i}\omega_0(t - t_0)}\sqrt{\frac{2m\omega_0}{\hbar}}\right], \tag{100}$$

and reduces to the coherent state (47) at the initial time $t = t_0$. This function is normalized to unity and is an eigenfunction of the invariant a with eigenvalue α.

From the boson invariant (99) we can form the quadratic invariant $I = \hbar\omega_0\{a, a^\dagger\}/2$ which has eigenvalues $\hbar\omega_0(n + 1/2)$ and for which the eigenfunctions and hence solutions of the TDSE take the familiar form $\psi_n(x, t) = \langle x|(a^\dagger)^n|0, t\rangle$ where the time-dependent vacuum is given by (100) with $\alpha = 0$, and the boson invariant $a^\dagger$ is the Hermitean conjugate of (99). We may determine these solutions directly by means of $U(t, t_0)$ as given in (97), which we re-order using the BCH formula (77) in which we choose the dilatation function

$$\rho^2 = 1 + \omega_0^2(t - t_0)^2, \tag{101}$$

which satisfies the Ermakov equation (25) with $\omega(t) = 0$. The function τ defined by (21) is given by $\tau = \arctan\omega_0(t - t_0)/\omega_0$, hence

$$\exp\left[-\frac{\mathrm{i}(t - t_0)p^2}{2m\hbar}\right] = \exp\left[\frac{\mathrm{i}m\dot\rho q^2}{2\hbar\rho}\right]\exp\left[-\frac{\mathrm{i}\log\rho(pq + qp)}{2\hbar}\right]\exp\left[-\frac{\mathrm{i}\tau H^0_{\mathrm{HO}}}{\hbar}\right],$$

where the time-independent harmonic oscillator Hamiltonian $H^0 = H^0_{\mathrm{HO}}$ is given by (24). Denote by $\psi_n(x)$ the eigenfunctions of H^0_{HO} (i.e., the functions (41) in which $\rho \to 1$ and omitting the time-dependent phase) then the solutions $\psi_n(x, t) = U(t, t_0)\psi_n(x)$ of the TDSE for the Hamiltonian (94), which are also eigenfunctions of $I = \hbar\omega_0\{a, a^\dagger\}/2$, are given by

$$\psi_n(x, t) = \mathrm{e}^{-\mathrm{i}m\gamma/\hbar}\,\mathrm{e}^{\mathrm{i}m\dot\sigma x/\hbar}\,\mathrm{e}^{-\mathrm{i}(n+\frac12)\omega_0\tau}\exp\left[\frac{\mathrm{i}m\dot\rho(x - \sigma)^2}{2\hbar\rho}\right]\psi_n\left(\frac{x - \sigma}{\rho}\right), \tag{102}$$

where σ, ρ are given by (96) and (101). The vacuum state $\psi_0(x, t)$ agrees with (100) at $\alpha = 0$ as follows from $1 + \mathrm{i}\omega_0(t - t_0) = \rho\,\mathrm{e}^{\mathrm{i}\omega_0\tau}$.

We may define $\mathfrak{su}(1,1)$ generators as in (50) and therefore also $\mathfrak{su}(1,1)$ coherent state solutions of the TDSE, as explained in section 4.4 for the harmonic oscillator. Barut–Girardello coherent states, defined as eigenfunctions of K_-, are given by the even and odd parts of the coherent states (100), regarded as functions of x. Perelomov coherent states are defined by (57) or (58) and the excited states are given by (65). As observed in section 4.4 these states are eigenfunctions of K_ζ^0 as defined by (61) and can be determined by performing the ζ-transformation (28) on the dilatation function ρ which satisfies the Ermakov equation (25). The special solution ρ in (101), which satisfies the Ermakov equation $\rho^3\ddot{\rho} = \omega_0^2$ with the initial conditions $\rho(t_0) = 1$, $\dot{\rho}(t_0) = 0$, leads by means of (28) to the following general solution ρ_ζ:

$$\rho_\zeta^2 = \frac{[1+\zeta+\mathrm{i}\omega_0(t-t_0)(1-\zeta)][1+\overline{\zeta}-\mathrm{i}\omega_0(t-t_0)(1-\overline{\zeta})]}{1-|\zeta|^2},$$

where $|\zeta| < 1$. The excited Perelomov states are eigenfunctions of the quadratic invariant $I = \hbar\omega_0\{a, a^\dagger\}/2$ in which we replace $\rho \to \rho_\zeta$, and hence the Perelomov coherent states $\psi_{n,\zeta}(x,t)$ are obtained by replacing $\rho \to \rho_\zeta$ in the functions $\psi_n(x,t)$ in (102), together with $\tau \to \tau_\zeta$ where τ_ζ is defined by (31) and may be calculated using (30). An alternative parametrization of the general solution ρ_ζ in terms of real parameters a, b is shown in (19) and leads to simpler expressions. Because these $\mathfrak{su}(1,1)$ coherent states evolve unitarily according to the TDSE they have the same overlap properties at equal times as the initial time-independent states.

In general, time-dependent translations allow static systems to be set into motion in which the Hamiltonian is modified by means of a translated potential and additional time-dependent terms which are linear in position.

6. Rotations

Having considered dilatations and translations we now investigate the case where T includes rotations in dimension $d = 2, 3$. If H^0 is as given in (10) then the corresponding Hamiltonian $\mathcal{H}$ in (5) can be arranged in the form of the Hamiltonian $\mathcal{H}_{\mathrm{em}}$ for an electromagnetic system:

$$\mathcal{H}_{\mathrm{em}} = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + e\Phi, \tag{103}$$

where $\boldsymbol{A}$, Φ are time-dependent vector and scalar potentials. Dynamical invariants when the scalar potential Φ is quadratic in r and solutions of the corresponding TDSE have been investigated by Lewis and Riesenfeld [3], also in [4, 127]. Solutions in which the Hamiltonian $\mathcal{H}$ is a time-dependent linear combination of the rotation generators $\boldsymbol{J}$ only have been considered by Perelomov ([21], chapter 19) using coherent states and also in [128] (chapter 3) by means of unitary transformations. In this case T corresponds to the time evolution operator U and is an element of the rotation group. Other work for the electromagnetic system (103) with a time-dependent uniform magnetic field is that of Dodonov *et al* [88] which investigates linear invariants, also [116, 129–132].

Here we apply the construction described in section 2, firstly for $d = 2$, where T consists of rotations and dilatations with H^0 of the form (10), and find a dynamical invariant and solutions of the corresponding TDSE for the Hamiltonian (103) for a uniform time-dependent magnetic field, with the scalar potential of a specified form. We generalize this to $d = 3$ and furthermore demonstrate that, by choosing H^0 to be a time-independent electromagnetic Hamiltonian, we may construct a dynamical invariant and solutions of the TDSE for an electromagnetic Hamiltonian in which the time-dependent magnetic and electric fields have a nontrivial spatial dependence.

6.1. Rotations in two dimensions

Rotations in two dimensions are performed by the unitary operator

$$T_{\mathrm{rot}}[\theta] = \exp\left[-\frac{\mathrm{i}}{\hbar}\theta(q_1p_2 - q_2p_1)\right] \tag{104}$$

where $\theta = \theta(t)$ is any time-dependent angle. For any function f we have $T_{\mathrm{rot}} f(\boldsymbol{x}) T_{\mathrm{rot}}^\dagger = f(\boldsymbol{x}_R)$ where the column vector $\boldsymbol{x}_R = R(\theta)\boldsymbol{x}$ denotes the time-dependent rotated coordinate vector, where the rotation matrix is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

For convenience let us include also dilatations, and define $T = T_{\mathrm{dil}} T_{\mathrm{rot}}$ where $T_{\mathrm{dil}}, T_{\mathrm{rot}}$ are defined by (11) and (104) respectively, and choose $H^0 = \boldsymbol{p}^2/2m + U(\boldsymbol{x})$ where the potential U need not be radial. Then the invariant is

$$I = TH^0T^\dagger = \frac{\rho^2\boldsymbol{p}^2}{2m} + U(\boldsymbol{x}_T), \tag{105}$$

where $\boldsymbol{x}_T = \boldsymbol{x}_R/\rho = R\boldsymbol{x}/\rho$ denotes the T-transformed coordinate vector. We also choose $F(I, t) = I/\rho^2$ then according to (5) the corresponding time-dependent Hamiltonian is

$$\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{\rho^2}U(\boldsymbol{x}_T) + \dot{\theta}(q_1p_2 - q_2p_1) + \frac{\dot{\rho}}{2\rho}(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}),$$

which takes the form of the electromagnetic Hamiltonian $\mathcal{H}_{\mathrm{em}}$ in (103) upon identifying

$$\boldsymbol{A}(\boldsymbol{x}, t) = -\frac{mc\dot{\rho}}{e\rho}\boldsymbol{x} - \frac{mc\dot{\theta}}{e}\boldsymbol{x}^\perp,$$

where $\boldsymbol{x}^\perp = (-x_2, x_1)$, and

$$\Phi(\boldsymbol{x}, t) = -\frac{m}{2e}\left(\frac{\dot{\rho}^2}{\rho^2} + \dot{\theta}^2\right)\boldsymbol{x}^2 + \frac{1}{e\rho^2}U(\boldsymbol{x}_T). \tag{106}$$

The uniform time-dependent magnetic field is given by $B = \partial_1 A_2 - \partial_2 A_1 = -2mc\dot{\theta}/e$ hence, given $B = B(t)$, we may determine $\theta = \theta(t)$. The electric field $\boldsymbol{E} = -\nabla\Phi - \partial\boldsymbol{A}/c\partial t$ has the expression

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{m}{e}\left(\frac{\ddot{\rho}}{\rho} + \dot{\theta}^2\right)\boldsymbol{x} + \frac{m}{e}\ddot{\theta}\boldsymbol{x}^\perp - \frac{1}{e\rho^2}\nabla U(\boldsymbol{x}_T),$$

from which, given $\boldsymbol{E}$, we also identify ρ. Solutions ψ_λ of the TDSE are given by the general formula (6), specifically

$$\psi_\lambda(\boldsymbol{x}, t) = \rho^{-d/2}\exp\left(-\frac{\mathrm{i}\lambda\tau}{\hbar}\right)\phi_\lambda^0\left(\frac{R\boldsymbol{x}}{\rho}\right), \tag{107}$$

with $d = 2$, where τ is defined by (21) and where $\phi_\lambda^0(\boldsymbol{x})$ denotes the eigenfunctions of H^0 with eigenvalues λ. The time evolution operator is given by (8):

$$U(t, t_0) = T_{\mathrm{dil}}T_{\mathrm{rot}}\exp\left(-\frac{\mathrm{i}\tau}{\hbar}H^0\right), \tag{108}$$

where we have chosen the initial conditions $\rho(t_0) = 1, \theta(t_0) = 0$, which ensure that $T(t_0) = \mathbb{I}$.

As an example, if $U(\boldsymbol{x})$ is radial and quadratic, corresponding to the isotropic harmonic oscillator for $d = 2$, then the scalar potential (106) is also radial and the eigenfunctions ϕ_λ^0 of H^0 are a product of angular functions and radial functions, depending on ℓ, which may be expressed in terms of Laguerre polynomials, as discussed in section 4. More generally, if $U(\boldsymbol{x})$ corresponds to the non-isotropic harmonic oscillator, for which Φ is no longer radial, the eigenfunctions ϕ^0_λ are a product of one-dimensional harmonic oscillator wavefunctions. In this case we write $H^0 = H^0_1 + H^0_2$ as the sum of commuting one-dimensional harmonic oscillator Hamiltonians H^0_i, with corresponding eigenfunctions $\phi^0_{n_i}(x_i)$ and eigenvalues $\lambda_i = \hbar\omega^0_i(n_i + 1/2)$ for $i = 1, 2$. We also have two commuting invariants $I_i = TH^0_iT^\dagger, i = 1, 2$ and, by choosing $F(I_1, I_2, t) = (I_1 + I_2)/\rho^2$, we may determine solutions ψ_{n_1,n_2} of the TDSE by applying the formula (6). Invariants other than (105) can also be constructed using the time evolution operator U, for example the two-component vector invariant $\boldsymbol{I} = U\boldsymbol{p}U^\dagger$ can be used to write the solution as a two-dimensional integral transform, similar to that for the harmonic oscillator as outlined in section 4.6.

We may generalize the two-dimensional system discussed here to allow for a magnetic field which is not spatially uniform, by choosing H^0 to be a time-independent electromagnetic Hamiltonian, however we investigate this case by proceeding directly to $d = 3$.

6.2. Rotations in three dimensions

The unitary operator T_{rot} which performs three-dimensional rotations may be parametrized by three angles, for example a parametrization by means of the Euler angles α, β, γ, which are functions of time, is

$$T_{\text{rot}}[\alpha, \beta, \gamma] = \mathrm{e}^{-\mathrm{i}\alpha J_3/\hbar}\,\mathrm{e}^{-\mathrm{i}\beta J_2/\hbar}\,\mathrm{e}^{-\mathrm{i}\gamma J_3/\hbar}, \tag{109}$$

where the generators $\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{p}$ satisfy $\boldsymbol{J} \times \boldsymbol{J} = \mathrm{i}\hbar\boldsymbol{J}$. We have

$$\mathrm{i}\hbar\frac{\partial T_{\text{rot}}}{\partial t}T^\dagger_{\text{rot}} = \boldsymbol{b}\cdot\boldsymbol{J}, \tag{110}$$

where $\boldsymbol{b} = (b_1, b_2, b_3)$ is given explicitly, for the Euler angle parametrization, by

$$\begin{aligned} b_1 &= -\dot\beta\sin\alpha + \dot\gamma\cos\alpha\sin\beta \\ b_2 &= \dot\beta\cos\alpha + \dot\gamma\sin\alpha\sin\beta \\ b_3 &= \dot\alpha + \dot\gamma\cos\beta. \end{aligned}$$

For a given vector $\boldsymbol{b}(t)$ we therefore integrate these equations in order to determine the angles α, β, γ and hence obtain the explicit operator T_{rot}. The rotated coordinate column vector $\boldsymbol{x}_R$ is given by $T_{\text{rot}}\boldsymbol{x}T^\dagger_{\text{rot}} = \boldsymbol{x}_R = R\boldsymbol{x}$, where R is the real 3×3 orthogonal time-dependent matrix with unit determinant corresponding to the rotation operator T_{rot}. In the Euler angle parametrization, R is given by the transpose of the right-hand side of (109) in which the generators $\boldsymbol{J}$ are replaced by their three-dimensional matrix representations $(J_i)_{jk} = -\mathrm{i}\hbar\varepsilon_{ijk}$. It follows from (110) that R satisfies $(R^t\dot R)_{ij} = \varepsilon_{ijk}b_k$, where R^t denotes the transpose (i.e., inverse) of R.

We now find exact solutions to the TDSE for the Hamiltonian $\mathcal{H}_{\text{em}}$ given by (103) where the time-dependent fields $\boldsymbol{A}, \Phi$ take a form as specified below. We begin with the time-independent Hamiltonian

$$H^0 = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}^0\right)^2 + e\Phi^0,$$

where the given fields $\boldsymbol{A}^0, \Phi^0$ are functions of position $\boldsymbol{x}$ only, which generalizes the previous choice for H^0 in (10). Following again the construction outlined in section 2, we choose the unitary operator $T = T_{\text{dil}}T_{\text{rot}}$ where $T_{\text{dil}}, T_{\text{rot}}$ are defined by (11) and (109) respectively and compute the invariant

$$I = TH^0T^\dagger = \frac{\rho^2}{2m}\left[\boldsymbol{p} - \frac{e}{c\rho}\boldsymbol{A}_R(\boldsymbol{x}_T)\right]^2 + e\Phi^0(\boldsymbol{x}_T),$$

where $\boldsymbol{x}_T = \boldsymbol{x}_R/\rho = R\boldsymbol{x}/\rho$, and where the time-dependent vector field $\boldsymbol{A}_R$ is defined by $\boldsymbol{A}_R = R^t \boldsymbol{A}^0$, where we regard $\boldsymbol{A}^0$ as a column vector. The time dependence of $\boldsymbol{A}_R(\boldsymbol{x}_T)$ arises therefore both from the elements of R^t and the argument $R\boldsymbol{x}/\rho$.

Upon choosing $F(I,t) = I/\rho^2$ we obtain the following Hamiltonian:

$$\mathcal{H} = \frac{1}{2m}\left[\boldsymbol{p} - \frac{e}{c\rho}\boldsymbol{A}_R(\boldsymbol{x}_T)\right]^2 + \frac{e}{\rho^2}\Phi^0(\boldsymbol{x}_T) + \boldsymbol{b}\cdot\boldsymbol{J} + \frac{\dot\rho}{2\rho}(\boldsymbol{p}\cdot\boldsymbol{q} + \boldsymbol{q}\cdot\boldsymbol{p}), \tag{111}$$

which can be cast into the form (103) of the electromagnetic Hamiltonian $\mathcal{H}_{\rm em}$ by identifying

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{\rho}\boldsymbol{A}_R(\boldsymbol{x}_T) - \frac{mc\dot\rho}{e\rho}\boldsymbol{x} - \frac{mc}{e}\boldsymbol{b}\times\boldsymbol{x}, \tag{112}$$

and

$$\begin{aligned}\Phi(\boldsymbol{x},t) = &-\frac{m}{2e}\left(\frac{\dot\rho^2}{\rho^2} + \boldsymbol{b}^2\right)\boldsymbol{x}^2 + \frac{m}{2e}(\boldsymbol{b}\cdot\boldsymbol{x})^2\\ &+ \frac{\dot\rho}{c\rho^2}\boldsymbol{x}\cdot\boldsymbol{A}_R(\boldsymbol{x}_T) + \frac{1}{c\rho}\boldsymbol{b}\times\boldsymbol{x}\cdot\boldsymbol{A}_R(\boldsymbol{x}_T) + \frac{1}{\rho^2}\Phi^0(\boldsymbol{x}_T).\end{aligned} \tag{113}$$

The corresponding magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ is given by

$$\boldsymbol{B}(\boldsymbol{x},t) = \frac{1}{\rho^2}\boldsymbol{B}_R(\boldsymbol{x}_T) - \frac{2mc}{e}\boldsymbol{b}, \tag{114}$$

where $\boldsymbol{B}_R = \nabla\times\boldsymbol{A}_R = R^t\boldsymbol{B}^0$ and $\boldsymbol{B}^0 = \nabla\times\boldsymbol{A}^0$ is the given magnetic field, but now with the argument $\boldsymbol{x}_T = R\boldsymbol{x}/\rho$, and where we have used $\det R = 1$. The static electric field is given by $\boldsymbol{E}^0 = -\nabla\Phi^0$, and the time-dependent electric field corresponding to the potentials $\boldsymbol{A}$, Φ is given by

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) = &\frac{m}{e}\left(\frac{\ddot\rho}{\rho} + \boldsymbol{b}^2\right)\boldsymbol{x} + \frac{m}{e}\dot{\boldsymbol{b}}\times\boldsymbol{x} - \frac{m(\boldsymbol{b}\cdot\boldsymbol{x})}{e}\boldsymbol{b}\\ &+ \frac{\boldsymbol{x}\cdot\boldsymbol{B}_R(\boldsymbol{x}_T)}{c\rho^2}\boldsymbol{b} - \frac{\boldsymbol{b}\cdot\boldsymbol{B}_R(\boldsymbol{x}_T)}{c\rho^2}\boldsymbol{x} - \frac{\dot\rho}{c\rho^3}\boldsymbol{x}\times\boldsymbol{B}_R(\boldsymbol{x}_T) + \frac{1}{\rho^2}\boldsymbol{E}_R(\boldsymbol{x}_T),\end{aligned} \tag{115}$$

where $\boldsymbol{E}_R = R^t\boldsymbol{E}^0$.

Hence, given time-dependent magnetic and electric fields of this form, we identify $\boldsymbol{b}$ and ρ as functions of time and explicitly determine the unitary operator $T = T_{\rm dil}T_{\rm rot}$. Solutions of the TDSE are then given by (107) with $d = 3$ and the time evolution operator $U(t,t_0)$ is given by (108).

As an example, consider the Dirac monopole [133] for which we choose the static vector potential

$$\boldsymbol{A}^0(\boldsymbol{x}) = \frac{g}{r(r+x_3)}(-x_2, x_1, 0)$$

with a corresponding magnetic field $\boldsymbol{B}^0(\boldsymbol{x}) = g\boldsymbol{x}/r^3$ and a scalar potential $\Phi^0 = 0$, describing a magnetic monopole of strength g located at the origin, carrying zero electric charge. The corresponding Schrödinger equation has been solved by Tamm [134], see also Fierz [135], to obtain the eigenfunctions $\phi^0_\lambda(\boldsymbol{x})$ and eigenvalues λ. We define the time-dependent vector and scalar potentials $\boldsymbol{A}$, Φ as given by (112) and (113), with magnetic and electric fields as in (114) and (115). We find

$$\boldsymbol{B}(\boldsymbol{x},t) = \frac{g}{r^3}\boldsymbol{x} - \frac{2mc}{e}\boldsymbol{b}$$

and

$$\boldsymbol{E}(\boldsymbol{x},t) = \frac{m}{e}\left(\frac{\ddot\rho}{\rho} + \boldsymbol{b}^2\right)\boldsymbol{x} + \frac{m}{e}\dot{\boldsymbol{b}}\times\boldsymbol{x} - \frac{m(\boldsymbol{b}\cdot\boldsymbol{x})}{e}\boldsymbol{b} + \frac{g}{cr}\boldsymbol{b} - \frac{g(\boldsymbol{b}\cdot\boldsymbol{x})}{cr^3}\boldsymbol{x}, \tag{116}$$

where the last two terms may be expressed as the gradient of $g\boldsymbol{b}\cdot\boldsymbol{x}/cr$. Although the magnetic field is time dependent, the monopole of strength g remains stationary at the origin, however the Dirac string rotates with time as determined by the matrix $R(t)$. The monopole can be set into motion by means of time-dependent translations as described in section 5. The monopole carries zero electric charge although the electric field is singular at the origin, at the location of the monopole. The general solution of the TDSE can in principle be obtained as an integral transform, as before for the harmonic oscillator in section 4.6, by choosing the initial wavefunction $\psi^0_{\boldsymbol{k}}(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ where $\boldsymbol{k}$ is the wave vector, and then computing the wavefunction $\psi_{\boldsymbol{k}}(\boldsymbol{x}, t) = U(t, t_0)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ at any later time t using the time evolution operator in (108). Any initial wavefunction may then be expressed as a Fourier transform and the solution $\psi(\boldsymbol{x}, t)$ of the TDSE appears as a three-dimensional integral transform similar to (80) for $d = 1$.

7. Conclusion

For systems in which the time-dependent Hamiltonian $\mathcal{H}$ takes the specified form (5), we have demonstrated that exact solutions of the corresponding TDSE can be expressed in terms of eigenfunctions of a time-independent operator H^0, and that the time evolution operator U can be expressed in the closed form (8). This construction is achieved by combining properties of unitary transformations and dynamical invariants and although well known for specific models, we have shown that the formalism applies to a wide range of models by considering spatial time-dependent unitary transformations comprising dilatations, translations and rotations. Solutions of the TDSE are obtained in various forms depending on the initial wavefunction, corresponding to different choices of invariants and includes various coherent states, plane wave solutions, distributional solutions and general solutions expressed as integral transforms. The formalism accommodates non-Hermitean Hamiltonians $\mathcal{H}$ constructed according to (5), provided the non-Hermitean operator H^0 has real eigenvalues.

For the harmonic oscillator we have found explicit radial solutions and Perelomov coherent state solutions, and have demonstrated an equivalence between these solutions by means of the ζ-transformation, and have also discussed various algebraic relations between the linear and quadratic invariants. We have investigated time-dependent rotations, which are relevant to Hamiltonians with electromagnetic interactions, and have found general solutions of the TDSE in which the time-dependent magnetic and electric fields, which take a specified form, need not be spatially uniform.

Acknowledgment

I gratefully acknowledge the support of the Larry Biedenharn Fellowship.

References

[1] Reed M and Simon B 1975 *Fourier Analysis, Self-Adjointness* (New York: Academic)
[2] Yajima K 1987 *Commun. Math. Phys.* **110** 415–26
[3] Lewis H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458–73
[4] Malkin I A, Man'ko V I and Trifonov D A 1970 *Phys. Rev.* D **2** 1371–85
[5] Truax D R 1981 *J. Math. Phys.* **22** 1959–64
Truax D R 1982 *J. Math. Phys.* **23** 43–54
[6] Ray J R 1982 *Phys. Rev.* A **26** 729–33
[7] Truscott W S 1993 *Phys. Rev. Lett.* **70** 1900–3
[8] Kaushal R S 1992 *Phys. Rev.* A **46** 2941–4

[9] Efthimiou C J and Spector D 1994 *Phys. Rev.* A **49** 2301–11
[10] Brown L S 1991 *Phys. Rev. Lett.* **66** 527–9
[11] Husimi K 1953 *Prog. Theor. Phys.* **9** 381–402
[12] Maamache M 1998 *J. Math. Phys.* **39** 161–9
[13] Pedrosa I A 1987 *J. Math. Phys.* **28** 2662–4
[14] Li F-L, Wang S J, Weiguny A and Lin D L 1994 *J. Phys. A: Math. Gen.* **27** 985–92
[15] Mostafazadeh A 1997 *Phys. Rev.* A **55** 4084–8
[16] Song D-Y 2000 *Phys. Rev.* A **62** 014103
[17] Vargas Martìnez J M and Moya-Cessa H 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S618–20
[18] Guasti M F and Moya-Cessa H 2003 *J. Phys. A: Math. Gen.* **36** 2069–76
[19] Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772–80
[20] Zhang W-M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867–927
[21] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[22] Wolf F and Korsch H J 1988 *Phys. Rev.* A **37** 1934–41
[23] Dattoli G, Richetta M and Torre A 1988 *Phys. Rev.* A **37** 2007–11
[24] Fernández F M 1989 *Phys. Rev.* A **40** 41–4
[25] Xu J-B, Qian T-Z and Gao X-C 1991 *Phys. Rev.* A **44** 1485–90
[26] Kaushal R S and Parashar D 1997 *Phys. Rev.* A **55** 2610–4
[27] Monteoliva D B, Korsch H J and Nuñez J A 1994 *J. Phys. A: Math. Gen.* **27** 6897–906
[28] Lai Y-Z, Liang J-Q, Müller-Kirsten H J W and Zhou J-G 1996 *Phys. Rev.* A **53** 3691–3
[29] Kim H-C, Lee M-H, Ji J-Y and Kim J K 1996 *Phys. Rev.* A **53** 3767–72
[30] Song D-Y 2003 *Phys. Rev.* A **68** 012108
[31] Shen J-Q, Zhu H-Y and Chen P 2003 *Eur. Phys. J.* D **23** 305–13
[32] Ray J R 1983 *Phys. Rev.* A **28** 2603–5
[33] Bender C M 2007 *Rep. Prog. Phys.* **70** 947–1018
[34] Hartley J G and Ray J R 1981 *Phys. Rev.* A **24** 2873–6
[35] Quesne C 1988 *J. Phys. A: Math. Gen.* **21** 3090–103
[36] Yaşuk F, Berkdemir C and Berkdemir A 2005 *J. Phys. A: Math. Gen.* **38** 6579–86
[37] Doescher S W and Rice M H 1969 *Am. J. Phys.* **37** 1246–9
[38] Munier A, Burgan J R, Feix M R and Fijalkow E 1981 *J. Math. Phys.* **22** 1219–23
[39] Berry M V and Klein G 1984 *J. Phys. A: Math. Gen.* **17** 1805–15
[40] Chetouani L, Guechi L and Hammann T F 1989 *Phys. Rev.* A **40** 1157–64
[41] Pinder D N 1990 *Am. J. Phys.* **58** 54–8
[42] Makowski A J and Dembinski S T 1991 *Phys. Lett.* A **154** 217–20
[43] Almeida R and Morales D A 1997 *Int. J. Quantum Chem.* **63** 827–33
[44] Pereshogin P and Pronin P 1991 *Phys. Lett.* A **156** 12–6
[45] Makowski A J 1992 *J. Phys. A: Math. Gen.* **25** 3419–26
[46] Makowski A J and Pepłowski P 1992 *Phys. Lett.* A **163** 143–51
[47] Dodonov V V, Klimov A B and Nikonov D E 1993 *J. Math. Phys.* **34** 3391–404
[48] Wagner M 1996 *Phys. Rev. Lett.* **76** 4010–3
[49] Zou J and Shao B 2000 *Int. J. Mod. Phys.* **14** 1059–65
[50] von Roos O 1983 *Phys. Rev.* B **27** 7547–52
[51] Arias de Saavedra F, Boronat J, Polls A and Fabrocini A 1994 *Phys. Rev.* B **50** 4248–51
[52] Dekar L, Chetouani L and Hammann T F 1999 *Phys. Rev.* A **59** 107–12
[53] Plastino A R, Rigo A, Casas M, Garcias F and Plastino A 1999 *Phys. Rev.* A **60** 4318–25
[54] Roy B and Roy P 2002 *J. Phys. A: Math. Gen.* **35** 3961–9
[55] Alhaidari A D 2002 *Phys. Rev.* A **66** 042116
[56] Koç R and Koca M 2003 *J. Phys. A: Math. Gen.* **36** 8105–12
[57] Ou Y C, Cao Z and Shen Q 2004 *J. Phys. A: Math. Gen.* **37** 4283–8
[58] Chen G and Chen Z 2004 *Phys. Lett.* A **331** 312–5
[59] Quesne C 2007 *J. Phys. A Math. Theor.* **40** 13107–19
[60] Dantas C M A, Pedrosa I A and Baseia B 1992 *Phys. Rev.* A **45** 1320–24
[61] Gao X-C, Xu J-B and Qian T-Z 1991 *Phys. Rev.* A **44** 7016–21
[62] Cho K H and Kim S P 1994 *J. Phys. A: Math. Gen.* **27** 1387–92
[63] Ji J-Y, Kim J K and Kim S P 1995 *Phys. Rev.* A **51** 4268–71
[64] Maamache M 1995 *Phys. Rev.* A **52** 936–40
Maamache M 1996 *J. Phys. A: Math. Gen.* **29** 2833–7
Maamache M 2000 *Phys. Rev.* A **61** 026102

[65] Pedrosa I A, Serra G P and Guedes I 1997 *Phys. Rev.* A **56** 4300–3
[66] Ciftja O 1999 *J. Phys. A: Math. Gen.* **32** 6385–9
[67] Kim S P and Page D N 2001 *Phys. Rev.* A **64** 012104
[68] Choi J R 2003 *Int. J. Theor. Phys.* **42** 853–61
[69] Fan K, Zheng Y, Ren W and Ding S 2007 *Int. J. Quantum Chem.* **107** 1355–66
[70] Ermakov V P 1880 *Univ. Izv. Kiev Ser. III* **9** 1–25
Ermakov V P 2008 *Appl. Anal. Discrete Math.* **2** 123–45 (Engl. Transl.)
[71] Milne E W 1930 *Phys. Rev.* **35** 863–7
[72] Pinney E 1950 *Proc. Am. Math. Soc.* **1** 681
[73] Lewis H R 1968 *J. Math. Phys.* **9** 1976–86
Lewis H R 1967 *Phys. Rev. Lett.* **18** 510–2
[74] Nunez J A, Bensch F and Korsch H J 1991 *J. Phys. A: Math. Gen.* **24** 2069–81
[75] Barbero G J F, Vergel D G and Villaseñor E J S 2006 *Phys. Rev.* D **74** 024003
[76] Combescure M 1986 *Ann. Inst. Henri Poincaré* **44** 293–314
[77] Dhara A K and Lawande S V 1984 *J. Phys. A: Math. Gen.* **17** 2423–31
[78] Burgan J R, Feix M R, Fijalkow E and Munier A 1979 *Phys. Lett.* A **74** 11–4
[79] Hartley J G and Ray J R 1982 *Phys. Rev.* A **25** 2388–90
[80] Szekeres P 2004 *A Course in Modern Mathematical Physics: Groups, Hilbert Space and Differential Geometry* (Cambridge: Cambridge University Press)
[81] Stillinger F H 1977 *J. Math. Phys.* **18** 1224–34
[82] Biedenharn L C and Louck J D 1981 *Racah-Wigner Algebra in Quantum Theory (Encyclopedia of Mathematics and its Applications vol 9)* (Reading, MA: Addison-Wesley)
[83] Abramowitz M and Stegun I 1965 *Handbook of Mathematical Functions* (New York: Dover)
[84] Yeon K H, Kim H J, Um C I, George T F and Pandey L N 1994 *Phys. Rev.* A **50** 1035–9
[85] Hartley J G and Ray J R 1982 *Phys. Rev.* D **25** 382–6
[86] Crosignani B, Di Porto P and Solimeno S 1968 *Phys. Lett.* A **28** 271–2
[87] Solimeno S, Di Porto P and Crosignani B 1969 *J. Math. Phys.* **10** 1922–8
[88] Dodonov V V, Malkin I A and Man'ko V I 1972 *Physica* **59** 241–56
[89] Dodonov V V and Man'ko V I 1979 *Phys. Rev.* A **20** 550–60
[90] Rajagopal A K and Marshall J T 1982 *Phys. Rev.* A **26** 2977–80
[91] Schubert M and Vogel W 1983 *Phys. Rev.* A **28** 3668
[92] Pedrosa I A 1987 *Phys. Rev.* D **36** 1279–80
[93] Jannussis A D and Bartzis B S 1988 *Phys. Lett.* A **129** 263–6
[94] Pedrosa I A 1997 *Phys. Rev.* A **55** 3219–21
[95] Ji J-Y, Kim J K, Kim S P and Soh K-S 1995 *Phys. Rev.* A **52** 3352–5
[96] Klauder J R and Skagerstam B S 1985 *Coherent States, Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[97] Kastrup H A 2003 *Fortschr. Phys.* **51** 975–1134
[98] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41–55
[99] Lohe M A and Thilagam A 2004 *J. Phys. A: Math. Gen.* **37** 6181–99
Lohe M A and Thilagam A 2005 *J. Phys. A: Math. Gen.* **38** 461–83
[100] Thilagam A and Lohe M A 2007 *J. Phys. A: Math. Theor.* **40** 10915–33
[101] Brif C, Vourdas A and Mann A 1996 *J. Phys. A: Math. Gen.* **29** 5873–85
[102] Vourdas A 2006 *J. Phys. A: Math. Gen.* **39** R65–R142
[103] Gur Y and Mann A 2005 *Phys. At. Nucl.* **68** 1700–8
[104] Hecht K T 1987 *The Vector Coherent State Method and Its Application to Problems of Higher Symmetries (Lect. Notes Phys.* vol 290*)* (Berlin: Springer)
[105] LeBlanc R and Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** 31–48
[106] Biedenharn L C and Lohe M A 1992 *Commun. Math. Phys* **146** 483–504
[107] Gerry C C and Kiefer J 1988 *Phys. Rev.* A **38** 191–6
[108] Gilmore R 1974 *J. Math. Phys.* **15** 2090–2
[109] Gilmore R 1974 *Lie Groups, Lie Algebras, and Some of Their Applications* (New York: Wiley)
[110] Truax D R 1985 *Phys. Rev.* D **31** 1988–91
[111] Rubinstein Z 1969 *A Course in Ordinary and Partial Differential Equations* (New York: Academic)
[112] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* (New York: Springer)
[113] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1954 *Tables of Integral Transforms* vol II (New York: McGraw-Hill)

[114] Boschi C D E, Ferrari L and Lewis H R 1999 *Phys. Rev.* A **61** 010101
[115] Myers V W 1960 *Am. J. Phys.* **28** 114–6
[116] Balasubramanian S 2001 *Am. J. Phys.* **69** 508–11
[117] Feng M 2001 *Phys. Rev.* A **64** 034101
[118] Guedes I 2001 *Phys. Rev.* A **63** 034102
Guedes I 2003 *Phys. Rev.* A **68** 016102
[119] Bauer J 2002 *Phys. Rev.* A **65** 036101
[120] Bekkar H, Benamira F and Maamache M 2003 *Phys. Rev.* A **68** 016101
[121] Liang M-L, Zhang Z-G and Zhong K-S 2004 *Phys. Scr.* **70** 98–100
Liang M-L, Zhang Z-G and Zhong K-S 2004 *Czech. J. Phys.* **54** 397–402
[122] Schulze-Halberg A 2005 *Found. Phys. Lett.* **18** 535–48
[123] Luan P-G and Tang C-S 2005 *Phys. Rev.* A **71** 014101
[124] Robinett R W 1996 *Am. J. Phys.* **64** 803–8
[125] Arrighini G P, Durante N L and Guidotti C 1996 *Am. J. Phys.* **64** 1036–41
[126] Rau A R P and Unnikrishnan K 1996 *Phys. Lett.* A **222** 304–8
[127] Malkin I A, Man'ko V I and Trifonov D A 1969 *Phys. Lett.* A **30** 414
[128] Bohm A, Mostafazadeh A, Koizumi H, Niu Q and Zwanziger J 2003 *The Geometric Phase in Quantum Systems* (Berlin: Springer)
[129] Abdalla M S 1988 *Phys. Rev.* A **37** 4026–9
[130] Baseia B, Mizrahi S S and Moussa M H Y 1992 *Phys. Rev.* A **46** 5885–9
[131] Lo C F 1992 *Phys. Rev.* A **45** 5262–5
[132] Bouguerra Y, Maamache M and Bounames A 2006 *Int. J. Theor. Phys.* **45** 1791–7
[133] Dirac P 1931 *Proc. R. Soc. Lond.* A **133** 60–72
[134] Tamm I 1931 *Z. Phys.* **71** 141–50
[135] Fierz M 1944 *Helv. Phys. Acta* **17** 27–34